Measuring the knot of degeneracies and the eigenvalue braids near a third-order exceptional point

Y. S. S. Patil\textsuperscript{1}, J. H"{o}ller\textsuperscript{1}, P. A. Henry\textsuperscript{2}, C. Guri\textsuperscript{1}, Y. Zhang\textsuperscript{1}, L. Jiang\textsuperscript{1}, N. Kralj\textsuperscript{1,3}, N. Read\textsuperscript{1,2,4}, J. G. E. Harris\textsuperscript{1,2,4}

\textsuperscript{1}Department of Physics, Yale University, New Haven, Connecticut 06520, USA
\textsuperscript{2}Department of Applied Physics, Yale University, New Haven, Connecticut 06520, USA
\textsuperscript{3}Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, 2100 Copenhagen, Denmark
\textsuperscript{4}Yale Quantum Institute, Yale University, New Haven, Connecticut 06520, USA

Corresponding authors: yogesh.patil@yale.edu, jack.harris@yale.edu

When a system of $N$ coupled oscillators is tuned by varying its external control parameters around a closed path (i.e., a “control loop”), the system’s spectrum of eigenfrequencies must return to itself. In Hermitian systems this return is trivial, as each eigenfrequency returns to its original value. However, in non-Hermitian systems, where the eigenfrequencies are complex, the spectrum may return to itself in a topologically non-trivial manner, a phenomenon known as spectral flow. The spectral flow is determined by how the control loop encircles degeneracies, and for $N = 2$ this relationship is well understood. Here we extend this description to arbitrary $N$. We show that control loops generically produce braids of eigenfrequencies, and that for $N > 2$ these braids’ non-commutativity reflects the non-trivial geometry of the space of degeneracies. We demonstrate these features experimentally for $N = 3$ using a cavity optomechanical system whose tunability gives access to a third-order exceptional point and all of the spectra in its neighborhood.

Introduction

A very wide range of physical systems are described by first-order differential equations of motion that are linear in the system’s coordinates. This includes classical systems near to mechanical equilibrium (e.g. coupled oscillators and linear wave systems), closed quantum systems, and open quantum systems that can be brought to Lindblad form. In these descriptions, the system’s state is an $N$-dimensional complex vector whose time evolution is generated by an $N \times N$ complex matrix $H$ (known as the “dynamical matrix”, “Hamiltonian”, or “Lindbladian”), which we take to be traceless without loss of generality.

The qualitative behavior of such a system depends on the form of $H$, which reflects the relevant symmetries and conservation laws. For example, in the quantum description of closed systems, $H$ is Hermitian. On the other hand, Newtonian mechanics and classical electromagnetism both
allow for linear elements with nonreciprocity, gain, and loss; as a result, the classical equations of motion for \(N\) coupled, weakly-damped oscillators (whose positions and momenta are encoded as \(N\) complex numbers) may have \(H\) of any form.

Recent years have seen considerable interest in features that distinguish non-Hermitian systems from their Hermitian counterparts, including non-orthogonal eigenvectors; complex eigenvalues; and a type of degeneracy, known as an exceptional point (EP), at which \(H\) is non-diagonalizable. In addition, non-Hermitian systems respond to small changes of \(H\) in a qualitatively different manner than Hermitian systems.\(^1,2,3\) These differences offer practical routes to new forms of control, sensing, and robustness, and have been explored in optics, microwaves, electronics, acoustics, optomechanics, and qubits.\(^4,5,6,7,8,9,10,11,12,13\)

Despite rapid progress, some fundamental aspects of non-Hermitian systems remain poorly understood. For example, when a system’s parameters are varied around a closed loop (with this “control loop” chosen so that the spectrum is non-degenerate throughout), the eigenvalues may move around one another in the complex plane. The way in which they do so, viewed topologically, is what we will describe below as “spectral flow”. It is determined by the manner in which the control loop encloses degeneracies; however, the specific relationship between the loop, the degeneracies, and the resulting spectral flow is well known only for some special cases.

One such case is \(N = 2\), for which the spectrum can be parametrized by a single complex number \(z\) so that the two eigenvalues \(\lambda_{\pm}\) are given by \(\pm z^{1/2}\). The properties of the square-root function ensure that varying \(z\) around a loop will exchange the \(\lambda_{\pm}\) if (and only if) the loop winds an odd number of times around the degeneracy at \(z = 0\) (this description holds whether or not \(H\) is diagonalizable at \(z = 0\)).\(^2,3\) This form of spectral flow has been observed in several physical systems.\(^12,13,14\)

For \(N > 2\), an equivalently simple and general picture of the relationship between the control loop, degeneracies, and spectral flow is less well known. To date, theoretical and experimental studies of spectral flow in systems with \(N > 2\) have focused on special cases in which \(H\) is constrained (or assumed to possess symmetries) or on numerical simulations of specific systems, rather than on a general description of the spectral flow.\(^15,16,17,18,19,20,21,22,23,24,25,26\)

**Spectral flow for arbitrary \(N\)**

The spectral flow for any \(N\) can be described by regarding the spectrum of \(H\) as an unordered set \(\lambda\) of \(N\) points in the complex plane. We take the parameters controlling \(\lambda\) to be the \(N - 1\) complex coefficients in \(p_H\), the characteristic polynomial of \(H\). These coefficients define the “control space” \(\mathcal{L}_N \cong \mathbb{C}^{N-1}\). They smoothly parameterize the space of spectra, and have simple expressions in terms of the elements of \(H\). \(\mathcal{L}_N\) can be partitioned into two subspaces \(\mathcal{V}_N\) and \(\mathcal{G}_N\), corresponding respectively to whether or not the spectrum is degenerate. \(\mathcal{V}_N\) consists of the points where \(D\), the discriminant of \(p_H\), is zero (see the SI); it has complex codimension 1 in \(\mathcal{L}_N\).

While \(\mathcal{L}_N\) is topologically trivial, this need not be the case for \(\mathcal{G}_N\) and \(\mathcal{V}_N\). To describe these two subspaces, we note that varying the control parameters along a smooth curve \(C \subset \mathcal{G}_N\) causes the
$N$ points comprising $\lambda$ to be smoothly transported in the complex plane. Throughout, we take $C$ to be a closed curve (or “loop”); in addition, we fix a basepoint in $G_N$ and consider only loops that start and end at that point. In this case, traversing a loop $C$ causes the initial spectrum (i.e., $\lambda$ at the basepoint) to be smoothly transported back to itself. Such an evolution of $N$ distinct points in the complex plane is called a braid of $N$ strands (see, e.g., Ref. [27]). We will say that two braids are isotopic to one another if one of them can be continuously deformed into the other, keeping its endpoints fixed and its strands non-intersecting during the deformation. Isotopy is an equivalence relation, so we can define isotopy equivalence classes of braids. We define the spectral flow produced by a control loop $C$ to be the isotopy equivalence class $b$ of the corresponding braid of eigenvalues. Braids with the common basepoint can be concatenated to produce another such braid, and with that operation the $b$s then form a group $B_N$, the Artin braid group. For $N \geq 2$, $B_N$ is an infinite group.

Two isotopic braids arise from two control loops $C_1, C_2$ that can be continuously deformed into each other within $G_N$, and hence each isotopy class $b$ of braids corresponds to a homotopy class of based loops $C \subset G_N$. Concatenation of control loops gives a group operation on the homotopy classes of control loops, which thus form a group called the fundamental group $\pi_1$ of the space $G_N$. It follows from this discussion that $\pi_1(G_N) \cong B_N$. Because $L_N$ is topologically trivial, the nontrivial $\pi_1(G_N) \cong B_N$ arises solely because the space $V_N$ (consisting of the points at which the spectrum is degenerate) was removed from $L_N$ to produce $G_N$, leaving a hole that control loops can wind around in various (homotopic) ways that correspond to the elements of $\pi_1$.

To give a more concrete picture of $G_N$ and $V_N$ (and the ways in which loops in the former may encircle the latter), we briefly consider the cases $N = 2$ and $N = 3$. For $N = 2$, the reasoning above returns the familiar result $G_2 \cong \mathbb{C}\setminus\{0\}$ (the complex plane without the origin). The fundamental group of this space, $\pi_1(G_2)$, is isomorphic to $B_2 \cong \mathbb{Z}$ (the group of integers under addition), reflecting the fact that each loop in $G_2$ belongs to the homotopy class determined by its winding number, and concatenating loops results in a new loop whose winding number is the sum of the winding numbers of the concatenated loops.

For $N = 3$, we have $G_3 \cong \mathbb{C}^2 - V_3$ and $\pi_1(\mathbb{C}^2 - V_3) \cong B_3$. From the equation $D = 0$ we show in the SI that $V_3$ is a connected hypersurface that includes a singular point at the origin $(0,0)$ corresponding to three-fold degeneracy; the rest of $V_3$ consists of the two-fold degeneracies. The two-fold degeneracies form the space $K \times \mathbb{R}_{>0}$, where $K$ is the trefoil knot and $\mathbb{R}_{>0}$ plays the role of the radial distance from the three-fold degeneracy. Therefore, if we identify $\mathbb{C}^2$ with $\mathbb{R}^4$, intersecting $V_3$ with a real hypersphere $S^3$ centered at the origin gives $K$. This structure is shown in Fig. 1, along with $C$s from different homotopy classes and the braids they produce, and agrees with the fact that $B_3$ is also the fundamental group of the complement of $K$ in $S^3$.

This description highlights two important features common to all non-Hermitian systems with $N > 2$, but absent in the well-studied case $N = 2$. The first is that the subspace $V_N$ formed by the degeneracies has a non-trivial geometry. The second is that this geometry makes loops in $G_N$ non-commutative (as $B_N$ is non-Abelian for $N > 2$). This rich behavior appears in such a ubiquitous physical system (coupled classical oscillators) because an eigenvalue spectrum consists of the roots of the polynomial $p_H$, and non-Hermitian systems can realize any complex polynomial as $p_H$. In the mathematical context of complex polynomial equations, the braid and
knot structures described here are well-known features of the relation between a polynomial’s coefficients and its roots.

In this paper, we provide an experimental demonstration of these two features. Specifically, we use a three-mode mechanical system in which $\mathbf{H}$ can be tuned by control parameters $\Psi$ that span $L_3$ and so provide access to a three-fold degeneracy and all of the spectra in its neighborhood. We measure spectra on a hypersurface that surrounds the three-fold degeneracy, and find that the two-fold degeneracies form a trefoil knot. We show that varying $\Psi$ around a loop produces an eigenvalue braid whose spectral flow is determined by how the loop encircles the trefoil knot, and we demonstrate braids that can be concatenated to produce any element of $B_3$. These measurements are consistent with the general reasoning given above, and are also in quantitative agreement with a microscopic model of the system.

**Optomechanical realization**

Figure 2A shows a schematic of the experiment, which uses three vibrational modes of a Si$_3$N$_4$ membrane with dimensions $1 \text{ mm} \times 1 \text{ mm} \times 50 \text{ nm}$. These modes’ bare eigenvalues (i.e., in the absence of optomechanical effects) are $\tilde{\lambda}^{(0)} = (\tilde{\lambda}_1^{(0)}, \tilde{\lambda}_2^{(0)}, \tilde{\lambda}_3^{(0)}) = 2\pi \times (352.243 - 2.2i, 557.217 - 1.9i, 704.837 - 1.8i) \text{ Hz}$, where the real (imaginary) parts give each mode’s oscillation frequency (amplitude damping rate). Frequencies related to the mechanical modes are denoted with a tilde when given in the lab frame, and without a tilde in the frame $\mathcal{R}$ described below.

The dynamical matrix $\mathbf{H}$ governing these modes is controlled using the dynamical back-action (DBA) effect of cavity optomechanics. The membrane is placed in an optical cavity with linewidth $\kappa/2\pi = 190 \text{ kHz}$, input coupling rate $\kappa_{\text{in}} = 0.267 \kappa$, and optomechanical coupling rates $g = (g_1, g_2, g_3) = 2\pi \times (0.198, 0.304, 0.300) \text{ Hz}$. Details of the apparatus are in the SI and Ref. [34].

The cavity is driven with three tones produced from a single laser (“control”, Fig. 2A). The DBA from each tone induces a complex-valued shift in each mechanical mode’s eigenvalue. In addition, each pair of tones gives rise to an intracavity beatnote, which induces a complex-valued coupling between pairs of modes whose frequency difference is comparable to the beatnote frequency. In the resolved sideband regime ($\kappa \ll \tilde{\omega}_1^{(0)}$, where $\tilde{\omega}_i^{(0)} \equiv \text{Re}(\tilde{\lambda}_i^{(0)})$) these shifts and couplings can be tuned over the complex plane by varying the tones’ powers $P_k$ and detunings $\Delta_k$ ($k \in \{1,2,3\}$). An expression for $\mathbf{H}$ in terms of $P_k$, $\Delta_k$, $\kappa$, $\kappa_{\text{in}}$, $g$, and $\tilde{\lambda}^{(0)}$ is given in the SI. For the experiments described here, the tones’ common detuning $\delta$ (Fig. 2B) is varied, while their relative detunings are fixed and chosen to produce beatnote frequencies close to the differences between the $\tilde{\omega}_i^{(0)}$.

The beatnote frequencies are also chosen so that there is a rotating frame $\mathcal{R}$ (defined in the SI) in which the dynamical matrix $\mathbf{H}$ is time-independent (in the rotating wave approximation), and in which the bare eigenvalues $\lambda^{(0)}$ are almost degenerate (their non-degeneracy in $\mathcal{R}$ is set by $\eta$ (Fig. 2B)). In $\mathcal{R}$, the experimental control parameters $\Psi = (\delta, P_1, P_2, P_3)$ can readily tune the system to a three-fold degeneracy; they also provide linearly independent control of the
coefficients of $p_H$ in the neighborhood of this degeneracy, and hence span $\mathcal{L}_3$, as described above. $H$ is otherwise unconstrained and possesses no particular symmetry. As a result, the degeneracies it accesses (of a given order) are of their most generic type: i.e., at an $m$-order degeneracy, the Jordan normal form of $H$ contains a Jordan block of dimension $m$ (we denote such a point as $EP_m$).

Thus, within $R$ the mechanical modes can be described by the equation of motion

$$\ddot{x}(t) = -iH(\Psi)x(t) + f(t)$$

Here $x(t) = (x_1(t), x_2(t), x_3(t))^T$ and $f(t) = (f_1(t), f_2(t), f_3(t))^T$ are the modes’ complex-valued amplitudes and the external forces driving them. While Eq. 1 is the generic equation of motion for any linear system, we emphasize the form of $H(\Psi)$ realized here: specifically, that the controls $\Psi$ completely and smoothly parametrize all of the complex eigenspectra in a neighborhood that includes $EP_3$.

Measuring the eigenvalue spectrum

We determine the modes’ eigenvalue spectrum $\lambda$ by measuring their mechanical susceptibility. This is accomplished using a second laser (“probe”, Fig. 2A) whose intensity modulation $\tilde{A}(t)$ (in the lab frame) exerts a force $\tilde{f}(t) \propto \tilde{A}(t)g$. The same laser also produces a heterodyne signal $\tilde{V}(t) \propto g \cdot \tilde{x}(t)$. The membrane is driven with a harmonic force ($\tilde{A}(t) \propto \cos(\tilde{\omega}_{AM} t)$) and its complex-valued response at the drive frequency $\tilde{V}(\tilde{\omega}_{AM})$ is recorded with a lock-in amplifier. Fig. 2C shows a typical measurement of $\tilde{V}(\tilde{\omega}_{AM})$ near each $\tilde{\lambda}_i^{(0)}$. Each band shows a peak that reflects contributions from all three mechanical modes. Qualitatively, in the peak near $\tilde{\omega}_{in}^{(0)}$ the $i$th mode responds directly to the drive, while the other two modes respond via the DBA-induced coupling.

As shown in the SI, $\tilde{V}(\tilde{\omega}_{AM})$ can be represented as the sum of nine Lorentzians whose (complex) resonance frequencies are determined by the $\lambda_i$ (the eigenvalues in $R$, which are used as fit parameters) together with the laser tones’ relative detunings (which are constant and known $a priori$). The Lorentzians’ amplitudes $s_{i,j}$ (denoting the contribution of the $j$th mode to the peak near $\tilde{\lambda}_i^{(0)}$) are also fit parameters. The resulting fit is shown as the black line in Fig. 2C. Details of the fitting are in the SI. In the remainder of this paper $\lambda$ is determined from data and fits similar to those in Fig. 2C.

Locating the $EP_3$ point

The system’s $EP_3$ is identified by measuring $\lambda(\Psi)$ over a range of $\Psi$ and converting each $\lambda$ to the quantity $d = |\lambda_1 - \lambda_2| + |\lambda_2 - \lambda_3| + |\lambda_3 - \lambda_1|$. Ideally $d$ would vanish at $EP_3$ as $\sim |\Psi - \Psi_{EP_3}|^{1/3}$ where $\Psi_{EP_3}$ is the value of $\Psi$ corresponding to $EP_3$. In practice, fluctuations in $\Psi$ convert this sharp cusp in $d(\Psi)$ to a broad minimum centered at $\Psi_{EP_3}$. Measurements of $d(\Psi)$ and comparison to theory are shown in the SI. These give $\Psi_{EP_3} = (2\pi \times 54(7) \text{ kHz}, 128(8) \mu W, 428(3) \mu W, 304(15) \mu W)$, in good agreement with the value calculated from the values of $\kappa, \kappa_{in}, g$, and $\tilde{\lambda}(0)$ given above.
Locating EP2 points

To study the behavior of the eigenvalue spectrum on a hypersurface surrounding EP3, we measured $\lambda$ on the boundary of a 4D hyperrectangle centered close to $\Psi_{EP3}$. Such a hypersurface is the union of eight 3D hyperrectangles. The specific hypersurface $\mathcal{S}$ used here bounds the 4D hyperrectangle defined by:

$$-10 \text{kHz} \leq \delta / 2\pi \leq 106 \text{kHz}, \ 22 \mu\text{W} \leq P_1 \leq 240 \mu\text{W}, \ 289 \mu\text{W} \leq P_2 \leq 675 \mu\text{W}, \ 78 \mu\text{W} \leq P_3 \leq 702 \mu\text{W}.$$  

Thus, each of the 3D “faces” of $\mathcal{S}$ corresponds to a constant value of one control parameter. Measurements were made by densely rastering $\Psi$ over 61 distinct 2D “sheets” within $\mathcal{S}$ (these sheets are shown in the SI).

Data from a typical 2D sheet is shown in Fig. 3. For each value of $\Psi$ (i.e., for each pixel in the sheet) the driven response $\bar{V}(\bar{\omega}_{AM})$ was measured and fit as in Fig. 2C. To locate the EP2 points in $\mathcal{S}$ we consider two quantities:

$$D = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2$$

and

$$E = (\text{det}[\mathbf{S}])^{-2}$$

where $\mathbf{S}$ is the matrix formed by the $s_{i,j}$. Both $D$ and $E$ vanish at EP2, and both exhibit a phase winding of $2\pi$ around EP2. However, they provide complementary information: first, vanishing $D$ reflects degeneracy of eigenvalues, while vanishing $E$ reflects degeneracy of eigenvectors (see the SI); second, $D$ and $E$ are derived from different aspects of the fits to $\bar{V}(\bar{\omega}_{AM})$, and so reflect partially independent features of the data.

Zeros and phase windings in $D$ and $E$ are identified algorithmically (see the SI) and are shown in Fig. 3 as cyan circles. These are the experimentally identified EP2 points (with locations $\Psi_{EP2}$) in $\mathcal{S}$. The same analysis is applied to all 61 sheets. The complete data set and details of the analysis are in the SI.

The EP2 knot

Figure 4 shows all the $\Psi_{EP2}$ identified in this way. For ease of visualization, they are depicted using two projections of $\mathcal{S}$, both of which generically preserve knot equivalence classes. Fig. 4A shows a stereographic projection, while Fig. 4B uses a projection isomorphic to the one in Fig. 4A, but which is more easily connected to the control parameters. Details of both projections are in the SI. In both Fig. 4A and 4B, the experimentally identified EP2’s are seen to trace out a curve that forms a trefoil knot $\mathcal{K}$.

The structure of $\mathcal{K}$ can be further clarified by noting that the quantity $\theta$ (which is derived from $\lambda$ as described in the SI) is expected to vary monotonically from $-\pi$ to $+\pi$ along $\mathcal{K}$, and so provides a natural coordinate on the knot. To demonstrate this, each EP2 point in Fig. 4 is colored according to the value of $\theta$ measured at the corresponding $\Psi_{EP2}$.

The solid curve in Fig. 4 shows a fit of the measured $\Psi_{EP2}$ to the values calculated from standard optomechanics theory using $g$ and $\kappa$ as fitting parameters. The best-fit values are $g = 2\pi \times \{0.1979, 0.3442, 0.3092\}$ Hz and $\kappa = 2\pi \times 173.84$ kHz (details of the fitting are in the SI). These values are also used to generate the plots of $D$ and $E$ (labelled “theory”) in the bottom row of Fig. 3. The values of $g$ and $\kappa$ extracted by fitting the knot $\mathcal{K}$ in the three-mode spectrum (i.e., in Fig. 4) agree well with the values determined independently from measurements of the DBA.
Non-commuting eigenvalue braids

When $\Psi$ is varied around a loop $C$, $\lambda(\Psi)$ is expected to form a braid whose equivalence class $b$ is determined by the homotopy class of $C$ in $G_N$. To demonstrate this experimentally, we select a set of pixels from the 61 sheets described above that trace out three loops (with a common base point), as shown in Figs. 5A-C. The corresponding $\lambda(\Psi)$ for each loop is shown in Figs. 5D-F.

The three loops belong to different homotopy classes (as can be seen from Figs. 5A-C) and result in eigenvalue braids from $b = I, \sigma_1, \sigma_2\sigma_1$ (Figs. 5D-F, respectively). Since $\sigma_1$ and $\sigma_2\sigma_1$ together generate the group $B_3$, the loops in Fig. 5 can be concatenated to produce any braid of eigenvalues.

The correspondence between a loop’s homotopy class and the isotopy class of the braid it produces is a robust feature of the complete data set. This is illustrated in the SI, which shows braids produced by a number of other loops, as well as the non-commutation of concatenating loops.

Conclusions and Outlook

In summary, these results provide an experimental demonstration of the general relationship between control loops, degeneracies, and spectral flow in non-Hermitian systems. In particular, they highlight two qualitative features that are absent from the well-studied case $N = 2$, but which emerge for all $N > 2$. These are the non-trivial geometry of the space of degeneracies, and the non-commutation of concatenating control loops that encircle this space.

It is natural to ask what role the braids demonstrated here may play in the dynamics of a system. For example, if one eigenmode of the system is initially excited, and then the system is slowly evolved around a control loop, it might be expected that the excitation would remain in the eigenmode that is smoothly connected with the original one, in analogy with adiabatic (asymptotically slow) transport in Hermitian systems. If this were the case, a control loop would permute excitations among the normal modes, with the specific permutation determined by the loop’s homotopy class. Such a control scheme – where the outcome is determined by a topological property of the input – would be of considerable interest. However, in non-Hermitian systems adiabatic control loops do not transport excitations in this manner. Nevertheless, real-time loops that are not asymptotically slow have been shown to produce similar transport in special cases, and it remains an open question whether other control schemes (such as shortcuts to adiabaticity or tailored nonlinearities) can stabilize such transport more generally. Exploration of these possibilities may open new means for achieving robust topological control in oscillator systems.

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**Supplementary Materials**
Supplementary text
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References


Fig. 1. The trefoil knot of degeneracies and the eigenvalue braids for a three-mode system. (A) At a fixed distance from the three-fold degeneracy, the control space for the spectrum is $S^3$ (shown here in stereographic projection). The degeneracies in this space are all two-fold and form a trefoil knot (orange). Three control loops (green, red, blue), each parameterized by $0 \leq s \leq 1$ share a common basepoint (black cross). (B-D) Evolution of the eigenvalues as $s$ is varied around each loop in (A). The black crosses show $\lambda$ at the basepoint. The dashed lines are guides to the eye. This figure is calculated from the characteristic polynomial of a three-mode system (see the SI).
Fig. 2. Experimental schematic and measurement of the mechanical susceptibility. (A) Two lasers (red and blue paths) address two modes of an optical cavity (white) in a cryostat (gray). The cavity contains a Si$_3$N$_4$ membrane (yellow) whose three mechanical modes are shown in the yellow box. AOM: acousto-optic modulator; LO: local oscillator; LIA: lock-in amplifier; AM-in: amplitude modulation input. (B) The optical spectrum, showing the three control beams (green, light blue, orange) and the cavity mode (blue). The non-degeneracy of the bare modes in the frame $\mathcal{R}$ is set by $\eta = -2\pi \times 100$ Hz. The beams’ powers ($P_1, P_2, P_3$) and their common detuning ($\delta$) span the space of eigenspectra around $\text{EP}_3$. (C) The complex response $\hat{V}$ measured at frequencies $\tilde{\omega}_{\text{AM}}$ near $\tilde{\omega}_1(0)$ (top), $\tilde{\omega}_2(0)$ (center) and $\tilde{\omega}_3(0)$ (bottom). Here $\Psi = (2\pi \times 50 \text{ kHz}, 125 \mu W, 364 \mu W, 426 \mu W)$. The left column shows $|\hat{V}(\tilde{\omega}_{\text{AM}})|$ and the right column shows a parametric plot of $\hat{V}(\tilde{\omega}_{\text{AM}})$. The data points are colored by $\tilde{\omega}_{\text{AM}}$. A global fit (black lines) gives the system’s eigenvalues. The magnitude of each mode’s contribution (determined from the fit) is shown as the orange, green, and light blue curves in the left column.
Fig. 3. Locating EP$_2$ points in the neighborhood of EP$_3$. The complex-valued quantities $D$ and $E$ measured on a typical 2D sheet in the hypersurface $\mathcal{S}$. For this sheet, $P_3 = 78$ $\mu$W and $\delta = 2\pi \times 60$ kHz. Top row: raw data. Middle row: data after outlier rejection and smoothing (see the SI). Cyan circles: algorithmically identified $\Psi_{EP2}$. Bottom row: $D$ and $E$ calculated from optomechanics theory. Cyan squares: $\Psi_{EP2}$ determined from this calculation.
Fig. 4. The knot of EP$_2$ measured on a hypersurface enclosing EP$_3$. (A) All of the EP$_2$ locations ($\Psi_{EP2}$) shown in a stereographic projection of $\mathcal{S}$. The coordinates ($X,Y,Z$) are dimensionless combinations of the control parameters. (B) The data from (A) shown in a projection that linearly maps each “face” of $\mathcal{S}$ to one of the hexahedrons bounded by black lines (except for the face at $P_1 = 240 \mu W$, which does not contain any $\Psi_{EP2}$ and is mapped to the exterior of the plot). The solid curve is the best fit to the measured $\Psi_{EP2}$. The data and fit are both colored by $\theta$, which serves as a coordinate along the knot. Details of the projections, the fit, and $\theta$ are in the SI, as are animations of the data and fit.
**Fig. 5. Eigenvalue braids realized by control loops.** (A) – (C) Three control loops (green, red, blue) in $\mathcal{S}$, each from a different homotopy class and sharing a common basepoint (black sphere). The measured knot (yellow circles) and the best-fit knot (orange curve) are shown for reference. The projection is the same as in Fig. 4A. (D) – (F) The eigenvalue spectrum $\lambda(\Psi)$ as $\Psi$ is varied around the corresponding loop. $\xi$ indexes the values of $\Psi$ along each loop. The black crosses show $\lambda$ at the start and stop of the loop. The dashed lines are guides to the eye. The SI shows animated versions of these plots, and the theoretically predicted braids. It also shows braids produced by other loops, and the non-commutation of concatenating loops.
Supplementary material for “Measuring the knot of degeneracies and the eigenvalue braids near a third-order exceptional point”

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§1. Characteristic polynomial, discriminant, and the trefoil knot

For an $N \times N$ matrix $H$, the eigenvalues are the solutions of the characteristic equation
\[ \det(\lambda I - H) = 0, \]
which can be written as
\[ \lambda^N - a_1\lambda^{N-1} + a_2\lambda^{N-2} + \ldots + (-1)^Na_N = 0. \]

The coefficients $a_i$ are invariants of $H$ under similarity transforms (change of basis), and in particular $a_1 = \text{tr} \ H$ and $a_N = \det H$. The characteristic polynomial on the left-hand side of this equation can be factored as $\prod_{i=1}^N(\lambda - \lambda_i)$ where the roots $\lambda_i$ may be repeated. The coefficients $a_i$ are the elementary symmetric polynomials in the roots $\lambda_i$, namely $a_1 = \Sigma_{i=1}^N \lambda_i$, $a_2 = \Sigma_{i,j:i<j} \lambda_i \lambda_j$, \ldots, $a_N = \prod_{i=1}^N \lambda_i$.

The discriminant of the polynomial is defined as $D = \prod_{i<j}(\lambda_i - \lambda_j)^2$; it vanishes if any two roots of the polynomial are equal. Being a symmetric polynomial, it can be expressed algebraically in terms of the elementary symmetric polynomials $a_i$ (see e.g., Ref. [46]). The explicit expressions are simpler if we shift $H$ by a multiple of the identity so that $a_1 = \text{tr} \ H = 0$. Then for the quadratic, $N = 2$, the discriminant is the familiar expression $D = 4a_2$, and the roots are $\pm \sqrt{-a_2}$. Focusing hereafter on the cubic, $N = 3$, its discriminant is
\[ D = -4a_3^3 - 27a_3^2, \]
which enters the formulas for the roots.

Defining $x = a_3$, $y = -a_2$ as our coordinates in $\mathbb{C}^2$ (so that the characteristic polynomial is $p_H = \lambda^3 - y\lambda - x$), the solutions to the equation $D(x, y) = 4y^3 - 27x^2 = 0$ form an algebraic variety (a hypersurface) in $\mathbb{C}^2$ which has a singularity at $x = y = 0$. $D$ has a (weighted) scaling property, such that rescaling $x \rightarrow a\ x$, $y \rightarrow b\ y$, where $a$, $b$ are real and positive and $a^2 = b^3$, changes $D$ by a factor; this maps any nonzero solution to $D = 0$ to another. Thus the variety resembles a cone in $\mathbb{C}^2 \cong \mathbb{R}^4$, and it suffices to consider a cross section, that is the intersection of the variety with a hypersurface that (i) has the topology of a hypersphere $S^3$, (ii) surrounds the origin without passing through it or intersecting itself, and (iii) is everywhere transverse to the local action of an infinitesimal (say, by $a = 1 + \epsilon$, $\epsilon$ small) such scaling. Any two such hypersurfaces are isotopy equivalent (via a rescaling that depends on position on the hypersurface). A particular such hypersurface results from considering the unit hypersphere defined by $|x|^2 + |y|^2 = 1$. The points $(x, y)$ on the hypersphere that satisfy $D(x, y) = 0$ can be parameterized as $(x, y) = (r_x e^{3i\theta}, r_y e^{2i\theta})$, where $r_x$, $r_y$ are real positive constants, and $0 \leq \theta < 2\pi$ is a variable. These points lie on a 2-torus $T^2$ embedded in $S^3$, and form a closed curve which is a trefoil knot in this $S^3$.

For any knot or link in $\mathbb{R}^3$ or $S^3$, the fundamental group of its complement is an isotopy invariant of the knot or link called the knot group. The knot group of the trefoil is well-known to be the braid group $B_3$, or this can be inferred by reasoning as in the main text.
§2. Experimental setup

This section describes the optical and electronic schemes used in this work.

The optical cavity is a single-sided Fabry-Perot resonator that is 3.64 cm long, for which the free spectral range (FSR) is measured to be $\omega_{FSR} = 2\pi \times 4.12$ GHz. It is built with mirrors having 5 cm radius of curvature and nominal reflectivities 0.9998 and 0.99997. With the membrane placed approximately at the center of the cavity, the finesse is $2.37 \times 10^4$. Fitting the cavity’s reflection spectrum gives the input coupling efficiency $k_{in}/k = 0.267$. The optomechanical cavity is mounted in a vacuum can which is placed inside a cryostat maintained at temperature $T = 4.2$ K, though the cryogenic temperature does not play a central role in this work. Details of the device’s construction can be found in Refs. [49,50].

A detailed schematic of the extra-cavity setup is shown in Fig. S1. The optomechanical device is addressed by two Nd:YAG lasers (Innolight Prometheus), both operating with a wavelength of 1064 nm. The generation and use of the various optical tones derived from these lasers is described below. A schematic of the optical spectrum is depicted in Fig. S2.

§2.1 Laser 1 (“probe laser”)

Laser 1 is used to lock all of the optical tones relative to one of the cavity’s resonances. It is also used to drive the mechanical modes and to detect their motion.

A portion of the light from Laser 1 is frequency-shifted by 279.5 MHz to generate a “probe” tone, using two acousto-optic modulators in series (AOM1 [Gooch & Housego free-space AOBD] and AOM2 [Gooch & Housego Fibre-Q]) driven at 79.5 MHz and 200 MHz, respectively. The probe tone is locked to an optical resonance of the cavity at $\omega_{cav,1}$ using the Pound-Drever-Hall (PDH) technique. This is accomplished using an electro-optic modulator (EOM [New Focus 4001]) driven at 15 MHz. A lock bandwidth of 4 kHz is realized by using the output from PID1 (Liquid Instruments Moku:Lab) to tune the voltage-controlled oscillator (VCO) that drives AOM2. Applying the feedback to AOM2 effectively locks all of the beams’ detunings with respect to $\omega_{cav,1}$, as discussed in §2.2.

To prevent large-amplitude, low-frequency drifts from forcing AOM2 beyond its effective tuning range, PID2 (Mokulabs) applies feedback control (with 1 Hz bandwidth) to the tuning piezo inside Laser 1 to maintain the VCO frequency close to 200 MHz.

Intensity modulation of the probe beam at frequencies $\tilde{\omega}_{AM} \approx \tilde{\omega}_{1,2,3}$ is produced using the amplitude-modulation (AM) input of the function generator FG1 (HP 4682B) that drives AOM1. It is this intensity modulation that drives the membrane (via radiation pressure) for the susceptibility measurements (an example of which is shown in Fig. 2C of the main paper and discussed in §5 of this supplement). The modulation is sourced from a lock-in amplifier (LIA [Zurich Instruments, HF2LI]) and modulates the probe beam intensity with a depth $\sim 0.04$.

Another portion of the light from Laser 1, which does not pass through AOM1, serves as the “local oscillator” (LO) for the heterodyne measurements.

The probe, its PDH and AM sidebands, and the LO are combined and sent to the optomechanical cavity’s input port. These tones, together with the phase modulation sidebands
induced on them by the membrane’s motion, are then directed back from the optomechanical cavity through a circulator onto a photodetector (PD1 [Thorlabs PDA10CF, 150 MHz bandwidth]). The resulting photocurrent contains beatnotes at a variety of frequencies. To measure the beatnotes near 79.5 MHz, the output of PD1 is mixed down to 20.5 MHz (using a 100 MHz oscillator [Vaunix LSG121, not shown]), and then input to the LIA for phase-sensitive detection.

For all of the measurements described here, the probe power is ~ 15 μW and the LO power is ~ 1200 μW, as measured at PD1.

§2.2 Laser 2 (“control laser”)

Laser 2 is used to generate the three “control” tones (labeled 1, 2, 3 in Fig. 2B of the main text).

Laser 2 is locked to the unshifted light from Laser 1 with a frequency offset of 8234.098(3) MHz. This frequency is chosen so that Laser 2 addresses a cavity mode whose resonance frequency $\omega_{\text{cav},2}$ differs from $\omega_{\text{cav},1}$ by $2\omega_{\text{FSR}}$ (i.e., whose longitudinal mode number differs by two from the mode addressed by Laser 1). The motivation for this approach is described below.

The light from Laser 2 is then frequency-shifted to generate the control tones 1,2,3. This shift is achieved using two AOMs in series. The first is AOM3 (Gooch & Housego, AOBD), which is driven by FG2 (2 units of Rigol DG4162) at three frequencies: 79.5 MHz + $\omega_{1}^{(0)}$ + $\delta$, 79.5 MHz + $\omega_{2}^{(0)}$ + $\eta$ + $\delta$, and 79.5 MHz + $\omega_{3}^{(0)}$ + $\delta$ (where $\delta$ is the “common detuning” that serves as one of the components of $\Psi$). The second is AOM2 (which also controls the probe and LO), which is driven at 200 MHz.

As mentioned above, the PDH lock signal is applied to AOM2 to ensure that all of these beams track fluctuations of the cavity. As described in Refs. [49,50], the primary source of these fluctuations is low-frequency vibrations of the structure supporting the membrane chip. As a result, the two cavity modes will experience the same detuning only if they have the same optomechanical coupling to the membrane. To ensure that this is the case, the membrane’s position within the cavity is chosen so that it lies at a “sweet spot”, defined as a point where $\frac{d\omega_{\text{cav},1}}{dx_m}$ = $\frac{d\omega_{\text{cav},2}}{dx_m}$, with $x_m$ being the membrane’s position. The process for locating such a point is described in Refs. [54,48,50].

This also ensures that the optomechanical coupling between a given mechanical mode and any optical tone (whether produced by Laser 1 or Laser 2) is very nearly the same. In particular, the equality of the optomechanical coupling to the probe and control tones is useful in extracting the system’s eigenvalues (see §3, §4, §5).

The powers of the control tones are stabilized (with a bandwidth of 1 kHz) using PID3 (New Focus LB1005), which feeds back to a variable optical attenuator (VOA, Thorlabs V1000A). The control tone powers reported in this work are all measured at PD2 (Thorlabs PDA36A), and range from 0 μW to 800 μW.
§3. Optomechanical characterization

This section details the measurement of the optomechanical coupling rates $g$ and the bare mechanical eigenvalues $\tilde{\omega}^{(0)}$. Briefly, these are obtained by measuring the dynamical back-action (DBA) for each mechanical mode when the cavity is driven by a single control tone (i.e., rather than the three tones used in the main part of this work). Note that frequencies associated with the mechanical modes are written with a tilde (e.g., $\tilde{\lambda}$) when they are given in the lab frame, and without a tilde (e.g., $\lambda$) when they are given in the rotating frame $\mathcal{R}$ (which is defined in §4).

To characterize mode $i$ of the membrane (where the modes are indexed by $i \in \{1,2,3\}$), the optical cavity is driven by a single control tone with power $\sim 250 \mu W$ and detuning $\Delta$, which is stepped over a range $\sim 3\kappa$ centered near $-\tilde{\omega}^{(0)}_i$. At each value of $\Delta$, the mechanical susceptibility of the mode is measured (as described in §5) and fit to a complex Lorentzian to yield the resonance frequency $\tilde{\omega}_i \equiv \text{Re}(\tilde{\lambda}_i)$ and energy damping rate $\tilde{\gamma}_i \equiv -2\text{Im}(\tilde{\lambda}_i)$, both of which are tuned via dynamical back-action (DBA). This is illustrated in Fig. S3.

For $\Delta \ll -\tilde{\omega}^{(0)}_i$, the DBA is expected to approach zero. Thus, the asymptotes in Fig. S3 correspond to the bare mechanical eigenvalues $\tilde{\omega}^{(0)}$. In addition, the DBA-induced shift in $\tilde{\omega}_i$ is expected to be zero (and the shift in $\tilde{\gamma}_i$ is expected to be a maximum) at $\Delta \approx -\tilde{\omega}^{(0)}_i$. This feature is helpful in identifying any spurious shift $\Delta_0$ in the laser detuning. The value of $\Delta_0$ found from fitting data as in Fig. S3 is typically less than $2\pi \times 5 \text{ kHz}$. Once $\Delta_0$ is measured, it is compensated by adding a corresponding shift to the offset lock of Laser 2 (which generates the three control tones, see §2).

The best-fit values of $g$ and $\tilde{\lambda}^{(0)}$ are determined from a global fit of the dataset shown in Fig. S3 to standard optomechanics theory. These values are:

$$\tilde{\lambda}^{(0)} = 2\pi \times (352243.3^{0.1} - 2.2^{0.1}i, 557216.8^{0.1} - 1.9^{0.1}i, 704836.7^{0.1} - 1.8^{0.1}i) \text{ Hz}$$
$$g = 2\pi \times (0.198^{0.001}, 0.304^{0.001}, 0.300^{0.001}) \text{ Hz}$$

where the uncertainties indicate one standard deviation.

Note that we define $g = \sqrt{\eta_c} g_0$, where $g_0$ is the conventionally reported single-photon optomechanical coupling rate. Here the coupling efficiency $\eta_c = P_{\text{in}}/P_{\text{meas}}$, where $P_{\text{in}}$ is the optical power incident on the optomechanical device and $P_{\text{meas}}$ is the power detected at PD2. Since the absolute magnitude of $g_0$ is inconsequential to the results presented in this work, no attempt is made to calibrate $\eta_c$. However, for completeness we note that the theoretically expected value of $g_0 = 2\pi \times (5.5, 4.3, 3.9) \text{ Hz}$. 
§4. Optomechanical model of the three-mode system

This section describes the theoretical model for optically controlling $\mathbf{H}$, the dynamical matrix of the mechanical three-mode system. It also defines the rotating frame $\mathcal{R}$ in which the bare mechanical modes are nearly degenerate, and in which the EPs are accessed. Lastly, it details how the experimental parameters $\Psi$ span the space of spectra (i.e., of $\mathbf{H}$) near an EP$_3$.

The classical Hamiltonian function for the full optomechanical system (i.e., including the optical mode as well as the three mechanical modes) is:

$$\mathcal{H} = \hbar \left( \Omega_{\text{cav},z} - i \kappa / 2 \right) a^* a + \sum_{i=1}^{3} \hbar \bar{\nu}_i^{(0)} \bar{c}_i^* \bar{c}_i + \sum_{i=1}^{3} \hbar g_i \bar{c}_i^* \bar{c}_i a^* a$$

where $a$ is the complex amplitude of the optical mode addressed by the control laser, the $\bar{c}_i$ are the complex amplitudes of the three mechanical modes, and $^*$ denotes complex conjugation. The first two terms correspond to the uncoupled optical and mechanical oscillators, and the third term corresponds to their interaction (with coupling strengths $g_i$). Note that the expression for $\mathcal{H}$ includes the reduced Planck’s constant $\hbar$ only in order to conform with the broader literature on optomechanics, in which the coupling rates would be the single-photon rates and the mode amplitudes would be the corresponding quantum mechanical operators. Since this work is purely classical, the overall scale of $\mathcal{H}$ (and hence the appearance of $\hbar$) is irrelevant.

The dynamics of $a$ and $\bar{c}_i$ are governed by $\mathcal{H}$ via Hamilton’s equations. The optical cavity is also driven through its input port (with coupling $\kappa_{\text{in}}$) with a drive field

$$a_{\text{in}}(t) = \sum_{j=1}^{3} \sqrt{P_j / \hbar \Omega_j} e^{-i(\Omega_j t + \phi_j)}$$

corresponding to the three control tones with powers $P_j$, frequencies $\Omega_j$ (i.e. detunings $\Delta_j = \Omega_j - \omega_{\text{cav},z}$), and phases $\phi_j$. Because $\kappa \gg g_i$, the interaction term in $\mathcal{H}$ can be linearized with respect to $a$. And since $\kappa \gg \bar{y}_i$, the optical field can be adiabatically eliminated to yield an effective equation of motion for the mechanical modes, in which $a$ does not appear but in which the parameters of the cavity drive (i.e., of the control tones) do.

For the detunings used in this work (see Fig. 2B of the main text):

$$\Delta_1 = -\tilde{\omega}_1^{(0)} + \delta, \quad \Delta_2 = -\tilde{\omega}_2^{(0)} + \delta + \eta, \quad \Delta_3 = -\tilde{\omega}_3^{(0)} + \delta$$

(with $\eta = -2\pi \times 100$ Hz), the intensity beatnote between control tones $j$ and $k$ is at the frequency

$$|\Delta_{jk}| = |\Delta_j - \Delta_k| \approx |\tilde{\omega}_j^{(0)} - \tilde{\omega}_k^{(0)}|.$$
Under the rotating wave approximation, the control tones \(j\) and \(k\) thus couple only the two mechanical modes \(j\) and \(k\), and not the third mechanical mode. As a result, the equation of motion for the three mechanical modes reduces to the form

\[
\dot{\mathbf{c}} = -i \mathbf{\tilde{H}} \mathbf{c} = -i \mathbf{\tilde{H}} \mathbf{c}
\]

where

\[
\mathbf{\tilde{H}} = \begin{pmatrix}
\tilde{\lambda}_1^{(0)} & 0 & 0 \\
0 & \tilde{\lambda}_2^{(0)} & 0 \\
0 & 0 & \tilde{\lambda}_3^{(0)}
\end{pmatrix} + \begin{pmatrix}
\sigma_{11} & \sigma_{12} e^{i\Delta_{12} t} e^{i\phi_{12}} & \sigma_{13} e^{i\Delta_{13} t} e^{i\phi_{13}} \\
\sigma_{21} e^{-i\Delta_{12} t} e^{-i\phi_{12}} & \sigma_{22} & \sigma_{23} e^{i\Delta_{23} t} e^{i\phi_{23}} \\
\sigma_{31} e^{-i\Delta_{13} t} e^{-i\phi_{13}} & \sigma_{32} e^{-i\Delta_{23} t} e^{-i\phi_{23}} & \sigma_{33}
\end{pmatrix}
\]

and \(\phi_{jk} = \phi_j - \phi_k\).

The coefficients denoted by \(\sigma\) are time-independent, and depend on the parameters of the optical drive. Specifically, the off-diagonal components are given by:

\[
\sigma_{jk} = -i \kappa_{in} g_j g_k \left[ \frac{p_j}{\hbar \Omega_j} \frac{p_k}{\hbar \Omega_k} \chi_{cav}^* (\Delta_j) \chi_{cav} (\Delta_k) \left[ \chi_{cav} (\tilde{\omega}_j^{(0)} + \Delta_j) - \chi_{cav} (\tilde{\omega}_j^{(0)} - \Delta_k) \right] \right]
\]

and the diagonal components are given by:

\[
\sigma_{jj} = -i \kappa_{in} g_j^2 \sum_{k=1,2,3} \left[ \frac{p_k}{\hbar \Omega_k} |\chi_{cav} (\Delta_k)|^2 \left[ \chi_{cav} (\tilde{\omega}_k^{(0)} + \Delta_k) - \chi_{cav} (\tilde{\omega}_k^{(0)} - \Delta_k) \right] \right]
\]

where the cavity’s optical susceptibility is

\[
\chi_{cav} (\Delta) = \frac{1}{\kappa/2 - i\Delta}
\]

We can remove the explicit time dependence from \(\mathbf{\tilde{H}}\) by writing the equation of motion in the rotating frame \(\mathcal{R}\) defined by the transformation \(\mathbf{U}\) given immediately below. In this frame the equation of motion is

\[
\dot{\mathbf{c}} = -i \mathbf{H} \mathbf{c}
\]

where
\[ c = \begin{pmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{pmatrix} = U \begin{pmatrix} \tilde{c}_1(t) \\ \tilde{c}_2(t) \\ \tilde{c}_3(t) \end{pmatrix} = \begin{pmatrix} e^{i(\tilde{\omega}_1(t) + \eta)t} e^{-i\phi_1} & 0 & 0 \\ 0 & e^{i\tilde{\omega}_2(t)} e^{-i\phi_2} & 0 \\ 0 & 0 & e^{i(\tilde{\omega}_3(t) + \eta)t} e^{-i\phi_3} \end{pmatrix} \begin{pmatrix} \tilde{c}_1(t) \\ \tilde{c}_2(t) \\ \tilde{c}_3(t) \end{pmatrix} \]

and the dynamical matrix \( H = U \tilde{H} U^{-1} + i \tilde{\Omega} U^{-1} \) is time-independent:

\[
H = \begin{pmatrix} -\eta - i\tilde{\gamma}_1(0)/2 & 0 & 0 \\ 0 & -i\tilde{\gamma}_2(0)/2 & 0 \\ 0 & 0 & -\eta - i\tilde{\gamma}_3(0)/2 \end{pmatrix} + \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \tag{S1}
\]

It is in this frame \( \mathcal{R} \) that \( H \) can be brought to an EP\(_3\) degeneracy by controlling the second matrix in Eq. S1 (denoted \( \sigma \)) with \( \Psi = (\delta, P_1, P_2, P_3) \).

§4.1 Spanning the neighborhood of EP\(_3\)

To test whether the four experimental parameters \((\delta, P_1, P_2, P_3)\) span the space of spectra around an EP\(_3\) degeneracy, we use the inverse function theorem to argue about the existence of a map between these four parameters and the two complex coefficients \((x, y)\) of the characteristic polynomial of \( H_0 \) in the vicinity of \( \Psi_{\text{EP3}} \). For simplicity, here we use \( H_0 \) (the traceless version of \( H \), defined as \( H_0 = H - \text{tr}(H)I/3 \) where \( I \) is the identity matrix), in which case \( x = \text{det}(H_0) \) and \( y = \text{tr}(H_0^2)/2 \).

In particular, we consider the Jacobian \( J \) of this map, where

\[
J = \begin{pmatrix} \partial \text{Re}(x)/\partial \delta & \partial \text{Re}(x)/\partial P_1 & \partial \text{Re}(x)/\partial P_2 & \partial \text{Re}(x)/\partial P_3 \\ \partial \text{Im}(x)/\partial \delta & \partial \text{Im}(x)/\partial P_1 & \partial \text{Im}(x)/\partial P_2 & \partial \text{Im}(x)/\partial P_3 \\ \partial \text{Re}(y)/\partial \delta & \partial \text{Re}(y)/\partial P_1 & \partial \text{Re}(y)/\partial P_2 & \partial \text{Re}(y)/\partial P_3 \\ \partial \text{Im}(y)/\partial \delta & \partial \text{Im}(y)/\partial P_1 & \partial \text{Im}(y)/\partial P_2 & \partial \text{Im}(y)/\partial P_3 \end{pmatrix}
\]

and the derivatives are evaluated at \( \Psi_{\text{EP3}} \). \( J \) is continuously differentiable in \( \Psi \) because \( x \) and \( y \) are polynomials in the elements of \( H_0 \), which in turn are continuously differentiable in \( \Psi \) (over the range of \( \Psi \) used in these measurements). Therefore, if \( \text{det}(J) \neq 0 \), the parameters span the same space as \( x \) and \( y \) (which is the full space of spectra, as discussed in the main paper) in the neighborhood of the EP\(_3\).

Numerical evaluation of \( \text{det}(J) \) is carried out using the expression for \( H \) in Eq. S1 (and the location of \( \Psi_{\text{EP3}} \) as determined in §7), giving \( \text{det}(J) \approx 10^{30} (2\pi \text{ Hz})^9 / \text{W}^3 \). This value is non-zero. More precisely, it is of the order of magnitude expected from the form of \( J \). It is roughly equal to \((\lambda'(\text{typ}))^{10}/(\Delta P)^3(\Delta \delta)\) where \( \lambda'(\text{typ}) \) is the typical magnitude of the eigenvalues in the neighborhood of \( \Psi_{\text{EP3}} \), and \( \Delta P \) and \( \Delta \delta \) are the typical scales of the control parameters over which the \( \lambda_i \) vary.
§5. Extracting the spectrum from mechanical susceptibility measurements

This section describes the relationship between the system’s eigenvalue spectrum $\lambda$ and measurements of the mechanical susceptibility. In particular, it derives the functional form used to fit the susceptibility data (e.g., as shown in Fig. 2C of the main text).

In the rotating frame $\mathcal{R}$ (see §4), the mechanical modes’ response to a force $f(\omega)$ can be written in the Fourier domain as

$$c(\omega) = \chi(\omega)f(\omega)$$

where

$$\chi(\omega) = (\omega I - H)^{-1}$$

The principle behind the measurements used in this work is to apply a force $f(\omega)$, measure the mechanical response $c(\omega)$, and thus infer the susceptibility $\chi(\omega)$, which contains information about $\lambda$.

The measurement of the mechanical response is carried out in the lab frame, where

$$\bar{c}(\tilde{\omega}) = \begin{pmatrix} \bar{c}_1(\tilde{\omega}) \\ \bar{c}_2(\tilde{\omega}) \\ \bar{c}_3(\tilde{\omega}) \end{pmatrix} = \begin{pmatrix} c_1(\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta) \\ c_2(\tilde{\omega} - \tilde{\omega}_2^{(0)}) \\ c_3(\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{3} [\chi(\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta)]_{1,j} [f(\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta)]_j \\ \sum_{j=1}^{3} [\chi(\tilde{\omega} - \tilde{\omega}_2^{(0)})]_{2,j} [f(\tilde{\omega} - \tilde{\omega}_2^{(0)})]_j \\ \sum_{j=1}^{3} [\chi(\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta)]_{3,j} [f(\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta)]_j \end{pmatrix}$$

The force applied by the intensity modulation of the probe tone is (in the lab frame) $\tilde{f}(t) \propto e^{i\tilde{\omega}_{AM} t} g$, i.e. in the Fourier domain $\tilde{f}(\tilde{\omega}) \propto g \delta(\tilde{\omega} - \tilde{\omega}_{AM})$. In $\mathcal{R}$ this is:

$$f(\omega) \propto \begin{pmatrix} g_1 \delta(\omega - \tilde{\omega}_{AM} + \tilde{\omega}_1^{(0)} + \eta) \\ g_2 \delta(\omega - \tilde{\omega}_{AM} + \tilde{\omega}_2^{(0)}) \\ g_3 \delta(\omega - \tilde{\omega}_{AM} + \tilde{\omega}_3^{(0)} + \eta) \end{pmatrix}$$
Thus, driving the membrane with a single sinusoidal force results in motion at three different frequencies. However, the lock-in amplifier only detects motion at the drive frequency \( \tilde{\omega}_{AM} \), i.e.

\[
\tilde{V}[\tilde{\omega}_{AM}] = \alpha \int f_{LIA}(\tilde{\omega} - \tilde{\omega}_{AM}) \tilde{c}(\tilde{\omega}) \cdot g \, d\tilde{\omega} \approx \alpha \int_{\tilde{\omega}_{AM} - \xi}^{\tilde{\omega}_{AM} + \xi} \tilde{c}(\tilde{\omega}) \cdot g \, d\tilde{\omega}
\]

where \( f_{LIA}(x) \) is the filter function of the lock-in amplifier (which has effective bandwidth \( \xi \)), and \( \alpha \) is the transduction gain. As a result,

\[
\tilde{V}(\tilde{\omega}_{AM}) = \begin{cases} 
\alpha g_1^2 \left[ \chi(\tilde{\omega}_{AM} - \tilde{\omega}_1^{(0)} - \eta) \right]_{1,1} \equiv \tilde{V}_1(\tilde{\omega}_{AM}) & \text{for } \tilde{\omega}_{AM} \approx \tilde{\omega}_1^{(0)} \\
\alpha g_2^2 \left[ \chi(\tilde{\omega}_{AM} - \tilde{\omega}_2^{(0)}) \right]_{2,2} \equiv \tilde{V}_2(\tilde{\omega}_{AM}) & \text{for } \tilde{\omega}_{AM} \approx \tilde{\omega}_2^{(0)} \\
\alpha g_3^2 \left[ \chi(\tilde{\omega}_{AM} - \tilde{\omega}_3^{(0)} - \eta) \right]_{3,3} \equiv \tilde{V}_3(\tilde{\omega}_{AM}) & \text{for } \tilde{\omega}_{AM} \approx \tilde{\omega}_3^{(0)}
\end{cases}
\]

so that only the diagonal components of the susceptibility \( \chi(\omega) \) are measured. Each of these diagonal components contains \( \lambda \), so in principle it would suffice to measure \( \tilde{V}(\tilde{\omega}_{AM}) \) in just one of the frequency ranges (say, around \( \tilde{\omega}_1^{(0)} \)). However, to make the analysis robust against noise, \( \tilde{V}(\tilde{\omega}_{AM}) \) was measured in all three frequency ranges (i.e., around each of the \( \tilde{\omega}_1^{(0)}, \tilde{\omega}_2^{(0)}, \tilde{\omega}_3^{(0)} \)), and this nominally redundant data was fit to determine \( \lambda \).

To explicitly see the relation of the susceptibility \( \chi(\omega) \) to the eigenspectrum of \( H \), consider its diagonalization \( H = T D T^{-1} \), where

\[
D = \begin{pmatrix} 
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3 
\end{pmatrix}.
\]

It can be easily shown that \( \chi(\omega) = T(\omega I - D)^{-1}T^{-1} \), where \( (\omega I - D)^{-1} \) is diagonal and contains \( \lambda \) as

\[
(\omega I - D)^{-1} = \begin{pmatrix} 
\frac{1}{\omega - \lambda_1} & 0 & 0 \\
0 & \frac{1}{\omega - \lambda_2} & 0 \\
0 & 0 & \frac{1}{\omega - \lambda_3}
\end{pmatrix}.
\]
This can be used to write the $\tilde{V}_i(\tilde{\omega})$ explicitly in terms of $\lambda$ and the matrix elements of $T$ and $T^{-1}$ as

\[
\tilde{V}_1(\tilde{\omega}) = \alpha g_1^2 \left[ \frac{T_{11}(T^{-1})_{11}}{\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta - \lambda_1} + \frac{T_{12}(T^{-1})_{21}}{\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta - \lambda_2} + \frac{T_{13}(T^{-1})_{31}}{\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta - \lambda_3} \right] \]

\[
\tilde{V}_2(\tilde{\omega}) = \alpha g_2^2 \left[ \frac{T_{21}(T^{-1})_{12}}{\tilde{\omega} - \tilde{\omega}_2^{(0)} - \lambda_1} + \frac{T_{22}(T^{-1})_{22}}{\tilde{\omega} - \tilde{\omega}_2^{(0)} - \lambda_2} + \frac{T_{23}(T^{-1})_{32}}{\tilde{\omega} - \tilde{\omega}_2^{(0)} - \lambda_3} \right] \]

\[
\tilde{V}_3(\tilde{\omega}) = \alpha g_3^2 \left[ \frac{T_{31}(T^{-1})_{13}}{\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta - \lambda_1} + \frac{T_{32}(T^{-1})_{23}}{\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta - \lambda_2} + \frac{T_{33}(T^{-1})_{33}}{\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta - \lambda_3} \right] \]

To extract the eigenvalues $\lambda$, these three spectra can be fit to the sum of nine complex Lorentzians as

\[
\tilde{V}_1(\tilde{\omega}) = a_1 \left[ \frac{s_{11}}{\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta - \lambda_1} + \frac{s_{12}}{\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta - \lambda_2} + \frac{s_{13}}{\tilde{\omega} - \tilde{\omega}_1^{(0)} - \eta - \lambda_3} \right] + b_1
\]

\[
\tilde{V}_2(\tilde{\omega}) = a_2 \left[ \frac{s_{21}}{\tilde{\omega} - \tilde{\omega}_2^{(0)} - \lambda_1} + \frac{s_{22}}{\tilde{\omega} - \tilde{\omega}_2^{(0)} - \lambda_2} + \frac{s_{23}}{\tilde{\omega} - \tilde{\omega}_2^{(0)} - \lambda_3} \right] + b_2
\]

\[
\tilde{V}_3(\tilde{\omega}) = a_3 \left[ \frac{s_{31}}{\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta - \lambda_1} + \frac{s_{32}}{\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta - \lambda_2} + \frac{s_{33}}{\tilde{\omega} - \tilde{\omega}_3^{(0)} - \eta - \lambda_3} \right] + b_3
\]

where $a_i = \alpha g_i^2$ and $s_{ij} = T_{ij}(T^{-1})_{ji}$, and the three additional (complex) constants $b_i$ represent the lock-in-detection background. Of the 18 complex parameters in this model ($a_i$, $b_i$, $s_{ij}$, and $\lambda_i$), the amplitudes $s_{ij}$ are constrained by the fact that $TT^{-1} = I = T^{-1}T$, i.e. $\sum_j T_{ij}(T^{-1})_{ji} = 1 = \sum_j (T^{-1})_{ij}T_{ji}$, which implies that $\sum_j s_{ij} = 1 = \sum_j s_{ji}$. Therefore, the rows and columns of the matrix

\[
S = \begin{pmatrix}
    s_{11} & s_{12} & s_{13} \\
    s_{21} & s_{22} & s_{23} \\
    s_{31} & s_{32} & s_{33}
\end{pmatrix}
\]

each add to unity. These are five independent complex constraints, and are implemented in fitting the measured spectra as:
\[ s_{13} = 1 - s_{11} - s_{12} \]
\[ s_{23} = 1 - s_{21} - s_{22} \]
\[ s_{31} = 1 - s_{11} - s_{21} \]
\[ s_{32} = 1 - s_{12} - s_{22} \]
\[ s_{33} = s_{11} + s_{12} + s_{21} + s_{22} - 1. \]

In other words, the global fit of the measured spectra \( \tilde{V}_1[\tilde{\omega}] \), \( \tilde{V}_2[\tilde{\omega}] \), \( \tilde{V}_3[\tilde{\omega}] \) to nine complex Lorentzians is implemented with 13 complex fit parameters. The best-fit values for \( \lambda \) and \( S \) so extracted are used in various ways to identify the locations of EP2 and EP3 (see §7 – §11).
§6. Degeneracy of eigenvectors

This section describes the connection between eigenvector degeneracy and the vanishing of \( E = (\det(S))^{-2} \).

We first note that in the susceptibility measurements, the membrane motion is detected only at the actuation frequency \( \bar{\omega}_{AM} \), though the actuation induces motion at other frequencies via the intracavity intensity beatnotes (this point is discussed in §5). Thus, the only component of the force vector contributing to the detected motion when \( \bar{\omega}_{AM} \approx \bar{\omega}_{l}^{(0)} \) is \( f_{l} \), where we define \( f_{1} \propto (1 \hspace{1em} 0 \hspace{1em} 0)^{T}, \ f_{2} \propto (0 \hspace{1em} 1 \hspace{1em} 0)^{T}, \ f_{3} \propto (0 \hspace{1em} 0 \hspace{1em} 1)^{T} \). The motion induced by \( f_{l} \) is the sum of (three) Lorentzians with amplitudes \( s_{ij} \).

The amplitude \( s_{ij} \) is proportional to the product of the projection of the actuation force \( f_{l} \) onto the \( j \)th left and right eigenvectors of \( H \) because

\[
S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} T_{11}(T^{-1})_{11} & T_{12}(T^{-1})_{21} & T_{13}(T^{-1})_{31} \\ T_{21}(T^{-1})_{12} & T_{22}(T^{-1})_{22} & T_{23}(T^{-1})_{32} \\ T_{31}(T^{-1})_{13} & T_{32}(T^{-1})_{23} & T_{33}(T^{-1})_{33} \end{bmatrix}
\]

where \( T \) diagonalizes the dynamical matrix, i.e. \( H = TDT^{-1} \) (see §5). The columns of \( T \) are the right eigenvectors of \( H \) and the rows of \( T^{-1} \) are its left eigenvectors. In the basis of the system’s right eigenvectors, the force \( f_{l} \propto ((T^{-1})_{1i}, \ (T^{-1})_{2i}, \ (T^{-1})_{3i})^{T} \), and in the basis of the system’s left eigenvectors, \( f_{l}^{T} \propto (T_{i1}, \ T_{i2}, \ T_{i3}) \).

This understanding of \( s_{ij} \) in terms of projections of \( f_{l} \) along the system’s eigenbases allows one to draw conclusions about the behavior of \( \det(S) \) at EPs. At an EP, neither the left nor the right eigenvectors span the full space; therefore, at least two projections of a generic vector (like \( f_{l} \)) onto both the left and right eigenvectors must diverge. This implies that at least two columns of \( S \) diverge, and hence that \( \det(S) \) diverges and \( E = (\det(S))^{-2} \) vanishes.

Exceptions to this generic behavior occur in some special cases. For example, if \( H \) is at an EP and the degenerate eigenvectors are exactly parallel to one of the actuation forces \( f_{l} \), the projections do not diverge. However, realizing such a scenario would require fine-tuning all of the matrix elements of \( H \). This is not the case in the present work (as only the coefficients of the characteristic polynomial of \( H \) are tuned), and so our results are described by the generic scenario discussed in the preceding paragraphs.
§7. Locating the EP\textsubscript{3}

This section details the protocol for experimentally identifying the EP\textsubscript{3}.

We identify the value of control parameters (\(\Psi_{\text{EP3}}\)) that corresponds to EP\textsubscript{3} through the quantity \(d = |\lambda_1 - \lambda_2| + |\lambda_2 - \lambda_3| + |\lambda_3 - \lambda_1|\), which may be visualized as the perimeter of the triangle formed by the system’s three eigenvalues \(\lambda\) in the complex plane. At \(\Psi_{\text{EP3}}\) the three eigenvalues are equal, and so \(d = 0\).

The search for the EP\textsubscript{3} point starts with the estimate:

\[
\Psi_{\text{EP3}}^{\text{(thy)}} = (2\pi \times 49.7 \text{ kHz, } 115 \mu\text{W, } 387 \mu\text{W, } 285 \mu\text{W}).
\]

We proceed by fixing three of the control parameters to these values, and scanning the fourth (say, \(\Psi_t\)). At each value of \(\Psi\) in this 1D sweep, \(\lambda\) is measured (as described in §5) and converted to \(d(\Psi)\). The experimental estimate \(\Psi_{\text{EP3}}^{\text{(est)}}\) is then updated with the value of \(\Psi_t\) that minimizes \(d\) over that sweep. This process is then repeated for different choices of \(\Psi_t\), as shown in Fig. S4. The estimate resulting from these 1D sweeps is:

\[
\Psi_{\text{EP3}}^{\text{(est)}} = (2\pi \times 49.7 \text{ kHz, } 125 \mu\text{W, } 435 \mu\text{W, } 300 \mu\text{W}).
\]

To further refine the estimate of \(\Psi_{\text{EP3}}\), we then measure \(d(\Psi)\) on 2D sheets that pass through \(\Psi_{\text{EP3}}^{\text{(est)}}\). For each 2D sheet, two control parameters are scanned while the other two are fixed, resulting in a total of six sheets. The \(d(\Psi)\) measured on these sheets are shown in Fig. S5. For visualization purposes, Fig. S6 shows the same sheets, but arranged in 3D to illustrate that \(d(\Psi)\) is minimized in the neighborhood of \(\Psi_{\text{EP3}}^{\text{(est)}}\). In Figs. S5 and S6, the middle row shows the filtered data (see §9 for details of the filtering), and the bottom row shows the values of \(d(\Psi)\) calculated from \(H\) (§4) using the best-fit optomechanical parameters determined in §11.

Near to \(\Psi_{\text{EP3}}\) the quantity \(d(\Psi)\) is expected to scale as

\[d(\Psi) \sim |\Psi - \Psi_{\text{EP3}}|^{1/3},\]

but in practice the sharp cusp in \(d(\Psi)\) is broadened by fluctuations in \(\Psi\). Nevertheless, clear minima are visible in the measured \(d(\Psi)\), and their locations are given in Table S1 (see §9 for details of the algorithm used to identify the minima). The mean location of these minima is taken as the experimentally identified EP\textsubscript{3}:

\[
\Psi_{\text{EP3}}^{\text{(exp)}} = \left(\delta_{\text{EP3}}^{\text{(exp)}}, P_{1\text{EP3}}^{\text{(exp)}}, P_{2\text{EP3}}^{\text{(exp)}}, P_{3\text{EP3}}^{\text{(exp)}}\right)
= (2\pi \times 54(7) \text{ kHz, } 128(8) \mu\text{W, } 428(3) \mu\text{W, } 304(15) \mu\text{W})
\]

This compares well with the location of EP\textsubscript{3} that is obtained from the best-fit parameters returned by fitting the measured knot (see §11):

\[
\Psi_{\text{EP3}}^{\text{(knot)}} = (2\pi \times 60.2 \text{ kHz, } 116 \mu\text{W, } 477 \mu\text{W, } 329 \mu\text{W}).
\]
§8. Rastering the hypersurface $\mathcal{S}$

The hypersurface $\mathcal{S}$ described in the main text is the boundary of a 4D hyperrectangle, and is a union of eight 3D hyperrectangles, which we refer to as “faces”. Each of these 3D faces is spanned by three components of $\Psi$ (for example, $P_1, P_2, P_3$) which range from their minimum value to their maximum value (given below), while the remaining component of $\Psi$ (in this example it would be $\delta$) is held fixed at either its maximum or its minimum value. As a result, the 3D faces span the ranges:

\[-10 \text{ kHz} \leq \delta/2\pi \leq 106 \text{ kHz} \]
\[22 \mu\text{W} \leq P_1 \leq 240 \mu\text{W} \]
\[289 \mu\text{W} \leq P_2 \leq 675 \mu\text{W} \]
\[78 \mu\text{W} \leq P_3 \leq 702 \mu\text{W} \]

For ease of analysis, measurements of $\lambda$ were taken by densely rastering $\Psi$ within sixty-one 2D “sheets”, each lying within one of the eight 3D faces. The locations of these sheets are shown in Fig. S7, and the actual data sets (from all 61 sheets) are shown in Fig. S13.

As can be seen from Fig. S7, no 2D sheets lie within the two faces having constant $P_1$. This is because the optomechanical model (described in §4) predicts that the EP$_2$ lie only in the other six faces. The absence of EP$_2$ in the two faces with constant $P_1$ was confirmed by measuring $\lambda$ at several hundred locations in these two faces (not shown).
§9. Data analysis algorithms

Here we describe the algorithms used to locate the EPs in the 2D data sheets. We also describe the processing (outlier rejection and Gaussian filtering) applied to the data in these 2D sheets in order for the algorithms to perform effectively.

As described in the main text, each measurement of a mechanical spectrum (i.e., with the control parameters $\Psi$ set to a specific value) is fit to extract $\lambda$ and $S$ for this value of $\Psi$. These quantities are then converted into $d, D, E,$ and $t$ for this value of $\Psi$. The quantity $t$ is defined as:

$$t \equiv \frac{x}{y} = \frac{\hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3}{(\hat{\lambda}_1^2 + \hat{\lambda}_2^2 + \hat{\lambda}_3^2)}$$

where $x$ and $y$ are the coefficients of the characteristic polynomial (see §1 and §4.1). The eigenvalues of the traceless version of $H$ (see §4.1) are $\hat{\lambda}_j = \lambda_j - \bar{\lambda}$, where $\bar{\lambda} = (\lambda_1 + \lambda_2 + \lambda_3)/3$. As described below, $t$ is useful because its complex phase $\theta \equiv \text{Arg}(t)$ provides a coordinate along the trefoil knot.

Much of the analysis used in this study is based on densely rastering two components of $\Psi$, resulting in a 2D “sheet” in which $d(\Psi), D(\Psi), E(\Psi),$ and $t(\Psi)$ can be displayed. Analyzing the data in these sheets allows for the identification of EP2 and EP3 locations. It is usually straightforward to identify the EP locations from the measured $d(\Psi), D(\Psi), E(\Psi)$ as the vanishing or phase-winding of these quantities, which are readily evident in Fig. 3 of the main text and Fig. S13.

However, to apply a uniform approach to locating these points, we use a minima-finding algorithm and a vortex-finding algorithm. These algorithms can be adversely impacted by noise in the data and by occasional outlier data points. The noise we refer to is the apparently random pixel-to-pixel variations visible in the top row of Fig. 3 from the main text (i.e., superposed on the smooth behavior that is similar to the corresponding theory plot in the bottom row). The outliers we refer to are the few pixels whose value differs drastically from their neighbors in the top row of Fig. 3 from the main text. As a result, we apply outlier rejection followed by Gaussian filtering to each of the quantities $d(\Psi), D(\Psi), E(\Psi),$ and $t(\Psi)$, yielding the filtered versions $\tilde{d}(\Psi), \tilde{D}(\Psi), \tilde{E}(\Psi),$ and $\tilde{t}(\Psi)$, which are shown as the middle row in Fig. 3 of the main text and Figs. S5, S6, S13 of this supplement. For the complex quantities $D, E,$ and $t$, the real and imaginary parts are treated separately and then recombined.

A minima identification algorithm (described in §9.3) is applied to $\tilde{d}$ to locate the EP3 point as described in §7. Minima identification is also applied to the magnitudes of $\tilde{D}$ and $\tilde{E}$ to locate the EP2 points in the hypersurface $S$. A phase-vortex identification algorithm (described in §9.4) is applied to the arguments (i.e. complex phases) of $\tilde{D}$ and $\tilde{E}$, also to locate the EP2 points in $S$. Lastly, the argument of $\tilde{t}$ at each EP2 is the value of $\theta$ used to color the corresponding point in Fig. 4 of the main paper and Fig. S8.

The complete data set consisting of the sixty-one 2D sheets used to search for EP2 points in the hypersurface $S$ is shown in Fig. S13. The data set of six 2D sheets used to identify the EP3 point is shown in Fig. S5 and Fig. S6.
§9.1 Outlier rejection

Outliers were identified using a Tukey Fence, which tags a data point at $\Psi$ as an outlier if the value at $\Psi$ is outside the range \[ \{Q_1 - s \times (Q_3 - Q_1), Q_3 + s \times (Q_3 - Q_1)\}, \]
where the first and third quartiles $Q_1$ and $Q_3$ are defined over a 5 pixel $\times$ 5 pixel neighborhood of $\Psi$ within the 2D sheet under consideration (the neighborhood is clipped for $\Psi$ near the sheet’s edge). To ensure that only extreme outliers are tagged, we set $s = 6$. By way of illustration, if the data were Gaussian distributed, this would correspond to tagging only values beyond 8.7 standard deviations.

All tagged outliers are inspected individually to eliminate the possibility of a false tag. The value of each outlier is replaced with the median of its 5 pixel $\times$ 5 pixel neighborhood. In the end, $\sim 200$ of the $\sim 27,000$ measurements ($\sim 0.8\%$) of $\lambda$ and $S$ on the hypersurface $S$ were rejected as outliers.

§9.2 Gaussian filtering

After the outlier rejection described above, the data in each 2D sheet is convolved with a 2D Gaussian kernel with HWHM = 1.87 pixels. The filter kernel is clipped (and re-normalized) as appropriate for pixels that lie near the edge of the sheet.

§9.3 Minima Identification

For any quantity $f(\Psi)$ (which may be $\ddot{\theta}(\Psi)$, $|\ddot{D}(\Psi)|$, or $|\ddot{E}(\Psi)|$), a minimum is initially tagged at any value of $\Psi$ at which $f$ is the minimum over its 3 pixel $\times$ 3 pixel neighborhood. Since some of these initial tags are caused by noise, we only accept tags at which the magnitude of the second derivative is larger than a specific threshold. In particular, we require $|f''(\Psi)| > \zeta$, where the threshold $\zeta$ is chosen to be $\langle |f''| \rangle + 2\sigma_{|f''|}$, with the mean $\langle \cdots \rangle$ and standard deviation $\sigma$ evaluated over the entire data sheet.

The $\Psi$ that are tagged in this way are reported as the experimentally identified minima. When this analysis is applied to $\ddot{D}(\Psi)$, the minima correspond to experimental estimates of the EP$_3$ location. When this analysis is applied to $|\ddot{D}(\Psi)|$ and $|\ddot{E}(\Psi)|$, the minima correspond to the experimentally identified EP$_2$ points. At each identified minimum of $|\ddot{D}|$ and $|\ddot{E}|$, the value of $\theta = \text{Arg}(\ddot{f})$ at that location is reported as the measured $\theta$ for that EP$_2$. These points are shown in Fig. S8A,C.

§9.4 Phase-Vortex Identification

For each phase function (Arg$[\ddot{D}]$ and Arg$[\ddot{E}]$), the algorithm starts with a location $\Psi$ within the sheet and then considers the closed counter-clockwise path defined by the eight nearest neighbors of $\Psi$. The point $\Psi$ is tagged as a phase vortex if the unwrapped phase along this closed path changes by $\pm 2\pi$. 
It sometimes happens that this approach tags several neighboring $\Psi$ as phase vortices. To determine whether this results from pixelation of the data, or because different portions of the knot actually intersect the sheet in close-by locations, we algorithmically cluster\textsuperscript{48} any adjacent points identified as phase vortices based on their value of $\theta$ (which serves to distinguish different parts of the knot from each other). For each cluster identified in this way, the mean value of $\Psi$ is reported as the experimentally identified phase vortex ($\text{EP}_2$). Also, the mean value of $\theta(\Psi)$ for each cluster is reported as the measured $\theta$ for that $\text{EP}_2$. These points are shown in Fig. S8B,D.

It should be noted that all the phase vortices identified in this work show a winding of $\pm 2\pi$ along the closed path constructed above. This is expected for $\text{Arg} (\Delta \Psi)$ and $\text{Arg} (\Delta E)$, as $\lambda_i(\Psi) \sim |\Psi - \Psi_{\text{EP}_2}|^{1/2}$ in the neighborhood of an $\text{EP}_2$ point at $\Psi_{\text{EP}_2}$\textsuperscript{1}. The proof of this is given in Ref. [48].

§9.5 Theory Plots of $\Delta$ and $E$

Eqn. S1 (see §4) can be numerically diagonalized at a given $\Psi$ to find the eigenvalues $\lambda(\Psi)$. Similarly, we calculate $S(\Psi)$ from the $T$-matrix associated with this diagonalization, cf. §5. The theoretical $\Delta(\Psi)$ and $E(\Psi)$ so evaluated are depicted in the bottom rows in Fig. 3 of the main text, and in Fig. S13.

The cyan squares in the theory plots of Fig. 3 mark the roots of $\Delta$ (corresponding to $\text{EP}_2$), which are found numerically. To make the numerical root-finding tractable, the $\text{EP}_2$ degeneracy of $H$ is cast as the system of equations

\[
\text{Re}[\Delta] = 0 \\
\text{Im}[\Delta] = 0
\]

where $\Delta$ is the discriminant of $p_H$.

In producing these theory plots, the parameters used are those obtained as the best-fit parameters for the knot data shown in Fig. 4 of the main text:

\[
\begin{align*}
g &= 2\pi \times (0.1979, 0.3442, 0.3092) \text{ Hz} \\
\kappa &= 2\pi \times 173.84 \text{ kHz}
\end{align*}
\]

The process by which the knot is fit is described in §11.
§10. Projections of the hypersurface $\mathcal{S}$

Here we describe the two projections that are used in Fig. 4 of the main text to represent data acquired on the hypersurface $\mathcal{S}$, which is the surface of a 4D hyperrectangle.

§10.1 Standard stereographic projection

Stereographic projection is a standard means for representing a sphere (typically of 1, 2 or 3 dimensions) in a Euclidean space with the same number of dimensions. In Fig. 4A of the main paper, we represent $\mathcal{S}$ by first projecting it onto the unit 3-sphere $\mathcal{S}^3$ and then applying the standard stereographic projection of $\mathcal{S}^3$ onto $\mathbb{R}^3$.

The map is constructed by first adimensionalizing the control parameter as

$$\psi' := \frac{\psi_{\text{exp}}}{\psi_{\text{EP3}}} - 1 := \left( \frac{\delta}{\delta_{\text{EP3}}} - 1, \frac{P_1}{P_{1,\text{EP3}}} - 1, \frac{P_2}{P_{2,\text{EP3}}} - 1, \frac{P_3}{P_{3,\text{EP3}}} - 1 \right)$$

and then normalizing it as $\psi'' := \frac{\psi'}{\|\psi\|}$, where $\| \cdot \|$ is the conventional $L^2$ norm. Note here the implicit use of the fact that $\psi_{\text{EP3}}$ lies inside the 4-volume bounded by $\mathcal{S}$.

Next, we act on $\psi''$ with a 4D rotation $\mathbf{R}$ (specified below). The new unit vector $\mathbf{R}\psi'' \equiv (x, z, w, y)$ is then stereographically projected onto the 3D cartesian coordinates $(X, Y, Z)$ as $X = \frac{x}{1-w}, Y = \frac{y}{1-w}, Z = \frac{z}{1-w}$. Thus, the choice of $\mathbf{R}$ corresponds to choosing the pole $(x, z, w, y) = (0, 0, 1, 0)$ for the stereographic projection.

The same pole is chosen for all the stereographic projections shown in this work (except for Fig. 1, see §12.1). It is chosen so as to optimize the visualization of the experimentally identified knot, and corresponds to $\psi'' = (0.1, -0.83, 0.55, 0)$, or equivalently, $\psi = (2\pi \times 55 \text{ kHz}, 22 \text{ \mu W}, 596 \text{ \mu W}, 304 \text{ \mu W})$.

§10.2 “Rectilinear stereographic” projection

The projection shown in Fig. 4B of the main text is isomorphic to the projection just described. However, it is intended to provide a more intuitive representation of the dimensionful experimental parameters $\psi$. Animations that describe this projection are given in Supplemental Movies S1 and S2.

This projection consists of five steps.

(i) We select one of the eight 3D hyperrectangles that constitute $\mathcal{S}$

(ii) We simply rescale its axes so that it forms a cube (this is the central cube in Fig. 4B)

(iii) Each of the six 3D hyperrectangles adjacent to the first one is also rescaled to form a cube, which is then attached to the first cube on their common 2D face. The resulting “six-way cross” faithfully represents the connections of the central cube to its six neighbors

(iv) To faithfully represent the connections among these six neighbors, a bilinear transformation is applied to each, deforming each cube into a truncated square pyramid. The transformation is chosen so that the 2D faces that are common to any two of these neighbors are
made to touch. These seven hexahedrons (the central cube and the six truncated square pyramids surrounding it) can readily be labelled by their original dimensionful axes, as in Fig. 4B. Together they form the bounding box of Fig. 4B.

(v) The final 3D hyperrectangle is mapped to the exterior of the bounding box via a nonlinear mapping, and extends to infinity (as does the standard stereographic projection described in §10.1).

As described in §8, there are no EP2’s in the two 3D hyperrectangles that correspond to constant $P_1$. We choose these to be the interior (cubical) hexahedron and the exterior region. This choice places all of the EP2’s in the six truncated square pyramids, facilitating a clear view of the knot. Supplemental Movies S2 and S3 give animated views of the data and fit shown in this projection.
§11. Fitting the EP\(_2\) locations to the optomechanical model

This section describes the fit of the three-mode optomechanical model to the 291 experimentally identified EP\(_2\) points shown in Fig. 4 of the main text. These locations are denoted as \(\Psi_{\text{EP}2}^{(\text{exp}, \ell)}\), with \(1 \leq \ell \leq 291\).

The best-fit parameters \(g\) and \(\kappa\) for the model are obtained by minimizing the cost function

\[
C(g, \kappa) = \sum_\ell \left| \Psi_{\text{EP}2}^{(\text{exp}, \ell)} - \Psi_{\text{EP}2}^{(\text{thy}, \ell)}(g, \kappa) \right|^2
\]

where the summands define a distance between the experiment and theory, which is adimensionalized by the EP\(_3\) coordinates \(\Psi_{\text{EP}3}^{(\text{exp})} = (\delta_{\text{EP}3}^{(\text{exp})}, p_{1,\text{EP}3}^{(\text{exp})}, p_{2,\text{EP}3}^{(\text{exp})}, p_{3,\text{EP}3}^{(\text{exp})})\).

In particular, for

\[
\Psi_{\text{EP}2}^{(\text{exp}, \ell)} = \left( \delta_{\text{EP}2}^{(\text{exp}, \ell)}, p_{1,\text{EP}2}^{(\text{exp}, \ell)}, p_{2,\text{EP}2}^{(\text{exp}, \ell)}, p_{3,\text{EP}2}^{(\text{exp}, \ell)} \right)
\]

and

\[
\Psi_{\text{EP}2}^{(\text{thy}, \ell)} = \left( \delta_{\text{EP}2}^{(\text{thy}, \ell)}, p_{1,\text{EP}2}^{(\text{thy}, \ell)}, p_{2,\text{EP}2}^{(\text{thy}, \ell)}, p_{3,\text{EP}2}^{(\text{thy}, \ell)} \right)
\]

this dimensionless distance (squared) is

\[
\left| \Psi_{\text{EP}2}^{(\text{exp}, \ell)} - \Psi_{\text{EP}2}^{(\text{thy}, \ell)}(g, \kappa) \right|^2
= \left( \frac{\delta_{\text{EP}2}^{(\text{exp}, \ell)} - \delta_{\text{EP}2}^{(\text{thy}, \ell)}}{\delta_{\text{EP}3}^{(\text{exp})}} \right)^2 + \left( \frac{p_{1,\text{EP}2}^{(\text{exp}, \ell)} - p_{1,\text{EP}2}^{(\text{thy}, \ell)}}{p_{1,\text{EP}3}^{(\text{exp})}} \right)^2 + \left( \frac{p_{2,\text{EP}2}^{(\text{exp}, \ell)} - p_{2,\text{EP}2}^{(\text{thy}, \ell)}}{p_{2,\text{EP}3}^{(\text{exp})}} \right)^2 + \left( \frac{p_{3,\text{EP}2}^{(\text{exp}, \ell)} - p_{3,\text{EP}2}^{(\text{thy}, \ell)}}{p_{3,\text{EP}3}^{(\text{exp})}} \right)^2
\]

Here, \(\Psi_{\text{EP}2}^{(\text{thy}, \ell)}(g, \kappa)\) is the EP\(_2\) point found numerically (as a root of the discriminant \(D(\Psi, g, \kappa)\), see §9) in a neighborhood of \(\Psi_{\text{EP}2}^{(\text{exp}, \ell)}\) and within its 2D data sheet. For example, if \(\Psi_{\text{EP}2}^{(\text{exp}, \ell)}\) is identified in a data sheet that rasters \(P_1\) and \(P_2\) while holding \(\delta\) and \(P_3\) fixed, the numerical root is found within the neighborhood

\[
\left( 0.65 P_{1,\text{EP}2}^{(\text{exp}, \ell)}, 1.35 P_{1,\text{EP}2}^{(\text{exp}, \ell)} \right) \times \left( 0.65 P_{2,\text{EP}2}^{(\text{exp}, \ell)}, 1.35 P_{2,\text{EP}2}^{(\text{exp}, \ell)} \right)
\]

at the same fixed values of \(\delta\) and \(P_3\). \(\Psi_{\text{EP}2}^{(\text{thy}, \ell)}(g, \kappa)\) is evaluated with \(\kappa_{\text{in}}/\kappa = 0.267\), and \(\tilde{\lambda}^{(0)}\) held equal to the values determined from the single-tone DBA measurements described in §2.

The minimization of \(C(g, \kappa)\) is implemented numerically on a high-performance cluster. The best-fit parameters so obtained are:
\[ g = 2\pi \times (0.1979, 0.3442, 0.3092) \text{ Hz} \]
\[ \kappa = 2\pi \times 173.84 \text{ kHz} \]

These parameters are used to produce the “best-fit knot” shown as the continuous curve in Fig. 4 in the main text. This curve is generated by using the best-fit values of \( g \) and \( \kappa \) given just above to calculate \( \lambda \) on 16,000 2D sheets in \( \mathcal{S} \). On each sheet, the EP\(_2\) points are identified as the roots of the discriminant \( D \) (found numerically as described in §9). At each of these EP\(_2\) points, \( \theta \) is also calculated. Finally, these points are colored according to \( \theta \) and are connected by straight line segments.

The values of the parameters \( g \) and \( \kappa \) given just above are also used to generate the theory plots in Figs. 3 and 4 of the main text, and in Figs. S4, S5, S6, S8, S11, and S13 of this supplement.
§12. Visualizing the eigenvalue braids

Here we describe various aspects of representing the eigenvalue braids, both for the experimental data and the theoretical calculations.

§12.1 Producing figure 1 of the main paper

This subsection describes how Fig. 1 of the main paper was generated. We first describe the calculation of the trefoil knot of EP\(_2\) locations shown in Fig. 1A. Then we describe the specific form of the control loops in Fig. 1A, and the corresponding eigenvalue braids shown in Figs. 1B-D.

Figure 1 was generated by considering the two complex coefficients \(x\) and \(y\) of the characteristic polynomial \(p_\mathbf{H} = \lambda^3 - y\lambda - x\) for a traceless 3 \(\times\) 3 matrix. As described in the main text (and in §1), the space spanned by \(x\) and \(y\) may be viewed as \(\mathbb{R}^4\), with the Cartesian coordinates \((\text{Re}(x), \text{Im}(x), \text{Re}(y), \text{Im}(y))^T\) giving a smooth parametrization of all the eigenspectra in the neighborhood of EP\(_3\), which is found at \(x = y = 0\).

Figure 1A shows a representation of the unit hypersphere \(S^3\) centered at the origin \((0,0,0,0)^T\). Specifically, Fig. 1A shows \(S^3\) mapped by a stereographic projection (whose pole is located at \((-1,0,0,0)^T\)) to the space \(\mathbb{R}^3\) spanned by the Cartesian coordinates \((X, Y, Z)^T\) defined via

\[
\text{Re}(x) = \frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}, \quad \text{Im}(x) = \frac{2z}{1 + x^2 + y^2 + z^2}, \quad \text{Re}(y) = \frac{2x}{1 + x^2 + y^2 + z^2}, \quad \text{Im}(y) = \frac{2y}{1 + x^2 + y^2 + z^2}
\]

The space spanned by \((X, Y, Z)^T\) is the space shown in Fig. 1A.

The yellow curve in Fig. 1A is defined by two constraints:

\[
|x|^2 + |y|^2 = 1 \\
4y^3 - 27x^2 = 0
\]

The first constraint simply defines \(S^3\) and so is satisfied everywhere in the space shown in Fig. 1A. The second constraint corresponds to the vanishing of the discriminant of \(p_\mathbf{H}\) (which is \(D = 4y^3 - 27x^2\)). Since \(D\) vanishes just where two (or more) eigenvalues of \(\mathbf{H}\) are degenerate, and since EP\(_3\) is not in \(S^3\), the orange curve shows all of the EPs in \(S^3\), and all of these are EP\(_2\). This curve is a trefoil knot, as described in §1.

Each of the three loops (green, red, and blue) shown in Fig. 1A can be written in the coordinate system \((X, Y, Z)^T\) as:

\[
X(d, r, \theta, \phi, s) = ((d - r) \sin(\theta) + r \sin(\theta + 2\pi s)) \cos(\phi) \\
Y(d, r, \theta, \phi, s) = ((d - r) \sin(\theta) + r \sin(\theta + 2\pi s)) \sin(\phi) \\
Z(d, r, \theta, \phi, s) = (d - r) \cos(\theta) + r \cos(\theta + 2\pi s)
\]
where \( \{d, \theta, \phi\} \) denotes the basepoint of the loop in spherical polar coordinates, \( r \) is the loop’s radius, and \( s \in [0,1] \) parameterizes the position along the loop (i.e. \( s = 0 \) and \( s = 1 \) both correspond to the loop’s basepoint).

All three loops in Fig. 1A have the same base point at \((d = 2.5, \theta = \frac{5\pi}{12}, \phi = -\frac{\pi}{12})^T\), which is shown as the black cross. Loops from three distinct homotopy classes were realized by using \( r = 0.4, 0.74, 1.2 \) for the green, red, and blue loop respectively.

To display the eigenvalue braids produced by each of these loops (shown in Figs. 1B-D), we find the three roots of \( p_H \) for 101 values of \( s \) ranging from 0 to 1. For each value of \( s \), the three roots (which comprise the eigenspectrum \( \lambda \) of \( H \)) are plotted in the complex plane. Their evolution as a function of \( s \) is shown in Figs. 1B-D by stacking a copy of the complex plane for each value of \( s \). The black crosses highlight \( \lambda \) at \( s = 0 \) (the bottom of each plot), which by construction is identical to \( \lambda \) at \( s = 1 \) (the top of each plot).

**§12.2 Producing figure 5 of the main paper**

The three control loops shown in Fig. 5A-C of the main text were assembled from data taken in two of the sixty-one 2D sheets. In Fig. 5A-C of the main paper, these loops are shown in the stereographic projection of the hypersurface \( S \) (described in §10.1) in order to highlight their relationship with the knot of EP2.

In Fig. S9, we show how these three loops lie within their 2D sheets. In Fig. S9, each gray circle represents a value of \( \Psi \) at which \( \lambda \) is measured (i.e., a “pixel” in the 2D sheets of Fig. S13). The green, red, and blue rectangles show the control loops that are produced by selecting (respectively) 21, 59, and 123 of these pixels. For each of these loops, the discrete variable \( \xi \) indexes the pixels along the loop (e.g., \( 1 \leq \xi \leq 59 \) for the red loop).

To compare the measured braids with theory, Fig. S11A-F shows the same panels as in Fig. 5 of the main paper, but together with Figs. S11G-I, which are the \( \lambda(\Psi) \) calculated using the best-fit parameters from the fit described in §11.

**§12.3 Additional experimental braids**

We emphasize that the experimental braids shown in Fig. 5D-F of the main paper are typical of the data described here. To illustrate this, Fig. S10 shows additional braids that are produced by more complicated control loops. The loops in Fig. S10 share a common basepoint, and realize the braids: \( \sigma_1^2 \) (Fig. S10D), \( \sigma_1^3 \) (Fig. S10E), and \( \sigma_1 \sigma_2 \sigma_1 \) (Fig. S10F).

**§12.4 Noncommutativity of the braids**

An important feature of the eigenvalue braids in systems with \( N > 2 \) is that they do not commute (hence, the group they form is non-Abelian). This is in contrast with the case \( N = 2 \), for which the braids do commute and so form an Abelian group. In principle, the noncommutativity of the measured braids is evident from their representation (i.e., in Fig. 5 of the main paper, and Figs. S10 and S11). However to explicitly demonstrate their noncommutativity, Fig. S12 shows the braids produced by concatenating two control loops, first
in one order and then in the opposite order. The two loops are shown in Fig. S12A in red and blue, and can be seen to belong to two different homotopy classes. Fig. S12B shows the evolution of the eigenvalues as \( \Psi \) is stepped first around the red loop and then around the blue loop (in the sense indicated by the arrows in Fig. S12A). The specific braid word produced is \( \sigma_1 \sigma_2 \). Fig. S12C shows the eigenvalues as \( \Psi \) is stepped first around the blue loop and then around the red loop (again in the sense indicated by the arrows in Fig. S12A). Here the braid word is \( \sigma_2 \sigma_1 \).

The inequivalence of the braids can be seen directly from the fact that the braids in (B) and (C) result in different permutations. For example, in (B) the strand that starts at the right-most \( \times \) (at \( \xi = 1 \)) evolves smoothly to the middle \( \times \) (at \( \xi = 116 \)), while in (C) the strand that starts at the right-most \( \times \) (at \( \xi = 1 \)) evolves smoothly to the left-most \( \times \) (at \( \xi = 116 \)). Thus, concatenating these two control loops in different order is seen to produce braids from different isotopy equivalence classes.

§12.5 Coloring the braid strands

Figures 5D-F of the main paper (as well as Figs. S10D-F and S12B,C) show experimentally measured braids. These are realized by stepping the parameters \( \Psi \) around the loops shown in Figs. 5A-C (and in Figs. S10A-C and S12A). At each value of \( \Psi \), the spectrum \( \lambda \) is determined from measurements as described in §5. The complete set of these measurements (i.e., at all of the values of \( \Psi \) around the loop) produces the data points shown in Fig. 5D-F of the main paper (and Figs. S10D-F and S12B,C).

However, coloring the individual strands (e.g., as light green, green, and dark green as in Fig. 5D) is potentially ambiguous. This is a consequence of the fact that \( \Psi \) is always stepped by a finite amount between measurements, while the components of \( \lambda \) at one value of \( \Psi \) are associated with specific components of \( \lambda \) at some other \( \Psi \) only via the fact that \( \lambda \) is a smooth function of \( \Psi \) (so long as \( \Psi \) is not an EP).\(^1\)

This means that if the steps in \( \Psi \) are sufficiently fine (and if the measurements of \( \lambda \) have little noise), then the braid strands’ identities will be evident from step to step. But if \( \Psi \) is stepped too coarsely (or if the measurements of \( \lambda \) are very noisy), then it will not be evident how to identify the strands from step to step.

It can be seen from Figs. 5D-F (as well as from the rotating versions of these figures, Supplemental Movie S4) that the steps in \( \Psi \) are sufficiently fine (and the noise in \( \lambda \) sufficiently small) that it would be straightforward to connect the measurements of \( \lambda \) into three braids “by eye”. However, to avoid any potential ambiguity, we implemented this “coloring” of the strands using a simple algorithm. Specifically, with each increment of \( \xi \) (which indexes the position along the control loop) i.e., from \( \xi \) to \( \xi + 1 \), each component of \( \lambda(\xi + 1) \) is associated with a component of \( \lambda(\xi) \) such that the sum of the distances
\[ Q = \sum_{m=1}^{3} |\lambda_m(\xi + 1) - \lambda_m(\xi)|^2 \]

is minimized. More precisely, \( Q \) is minimized over the six possible choices for identifying the components of \( \lambda(\xi + 1) \) with those of \( \lambda(\xi) \). Repeating this process for every value of \( \xi \), the braids are tracked and colored as depicted in Fig. 5 of the main text and in Figs. S10-12.
§13. Relation of the present work to non-Hermitian band structure

We note that topics related to those described in this work have been considered in the context of non-Hermitian band structure (NHBS). However there are a number of qualitative differences between NHBS and the non-Hermitian oscillators considered here, both in the physical systems being described and in the mathematical concepts relevant to the description.

The physical system under consideration in NHBS is a wave propagating in an \( \ell \)-dimensional lattice (where \( \ell \) is typically 1, 2, or 3) that possesses a combination of non-reciprocity, gain, and loss. Propagation in such a lattice can be characterized by bands whose dispersion is given by the complex eigenvalues of a matrix (which plays the role of \( \mathbf{H} \) in the present paper). A central question in NHBS is how these eigenvalues depend upon the quasimomentum \( \mathbf{k} \) (whose vector components play the role of the control parameters considered in the present paper). Thus in NHBS, the number of control parameters is limited to \( \ell \), and the control space they span (the analog of \( \mathcal{L}_N \) in the present work) is topologically non-trivial by assumption (since the Brillouin zone is an \( \ell \)-torus).

In contrast, for non-Hermitian oscillators the control space (\( \mathcal{L}_N \)) is topologically trivial, and the number of controls (i.e., the dimensionality of the control space) is sufficient to span the full space of complex eigenspectra. These features result in the direct connection between non-Hermitian oscillators and general complex polynomials (described in the main text under “Spectral flow for arbitrary \( N \)”). In particular, they lead to the non-trivial structure of the degenerate subspace, which in turn establishes the correspondence between control loops and the non-Abelian group of eigenvalue braids.

Lastly, we note that experiments on NHBS to date have been limited to braids realized by two eigenvalues. Thus, they correspond to the \( N = 2 \) case, for which the subspace of degeneracies has a trivial geometry, and the group formed by the eigenvalue braids is Abelian. In contrast, for the \( N = 3 \) case explored in the experiments described here, the subspace of degeneracies has a non-trivial geometry, and the eigenvalue braids form a non-Abelian group.

Another approach to studying the propagation of linear excitations in non-Hermitian lattices is provided by gyroscopic metamaterials. However, these systems possess purely real eigenvalues (owing to the symplectic symmetry of their dynamical matrix), and so do not exhibit the behavior considered in this work.

**Movie S1: Laying out the hypersurface $\mathcal{S}$ in terms of the experimental parameters.**

The surface of a 4D hyperrectangle is a union of eight 3D hyperrectangles. These 3D hyperrectangles are connected to each other via their common 2D faces. As described in §10.2, the “rectilinear stereographic” projection (used in Fig. 4B of the main paper) “glues” those common 2D faces together in a way that is isomorphic to the standard stereographic projection. This movie illustrates the construction of the rectilinear stereographic projection from the eight 3D data sets.

**00:00 – 00:07** An arrangement of the eight 3D hyperrectangles is shown. Each hyperrectangle is labelled by its fixed control parameter (green text). The labelled arrows on the axes of each hyperrectangle indicate the remaining three control parameters which vary within their bounds, i.e.: $-10$ kHz $\leq \delta / 2\pi \leq 106$ kHz, $22 \ \mu W \leq P_1 \leq 240 \ \mu W$, $289 \ \mu W \leq P_2 \leq 675 \ \mu W$, $78 \ \mu W \leq P_3 \leq 702 \ \mu W$.

**00:07 – 00:15** Some pairs of common 2D faces are chosen and highlighted in different colors.

**00:16 – 00:18** The highlighted faces are glued together by translating them towards the $P_1 = 22 \ \mu W$ hyperrectangle.

**00:18 – 00:24** The remaining hyperrectangles are glued on their common 2D faces. Then the six nearest neighbors of the $P_1 = 22 \ \mu W$ hyperrectangle are subject to a bilinear transformation that stretches them transverse to their $P_1$ axis. The eighth and final hyperrectangle ($P_1 = 240 \ \mu W$) is turned inside-out and its contents are stretched to infinity. The wireframe figure at 00:24 is the end result of the rectilinear stereographic projection.
**Movie S2: Visualizing the EP₂ knot in the rectilinear stereographic projection.** This video shows how the measured EP₂ locations (and the best-fit knot) appear in each of the eight 3D faces of the hypersurface $\mathcal{S}$. It also shows the smooth transformation of these faces (along with the data & fit) into the rectilinear stereographic projection of Fig. 4B from the main text.

00:00 – 00:05 The eight 3D faces of $\mathcal{S}$ are shown. They are arranged to form a “net” of the hypersurface $\mathcal{S}$. Also shown are the measured EP₂ locations (colored circles) and the best-fit knot (solid curve) from Fig. 4 of the main text. For each 3D face, the green text indicates the control parameter that is held constant.

00:05 – 00:16 The net is rotated to give a complete view. Note that the two 3D faces in which $P₁$ is held constant do not contain any EP₂ locations (in either the data or the fit).

00:16 – 00:26 The eight 3D faces, along with the data and fit inside them, are continuously deformed to realize the rectilinear stereographic projection. Note that the two empty 3D faces (i.e., the ones with constant $P₁$) are mapped to the innermost cube and to the region outside the frame. Thus, all of the EP₂ locations (in both the data and the fit) lie in the six hexahedrons that surround the central cube.

00:26 – 00:37 The rectilinear stereographic projection is rotated to give a complete view.

00:37 – 00:40 The axis labels are added.
**Movie S3:** This movie is simply a rotating version of Fig. 4 from the main text.
**Movie S4:** This movie is simply a rotating version of Fig. 5 from the main text.
### 15. Supplemental table S1

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<th>$P_2$ (µW)</th>
<th>$P_3$ (µW)</th>
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<tr>
<td>Std. Dev.</td>
<td>7.2</td>
<td>8</td>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

**Table S1**
Location of the minimum in $d$ for each of the six 2D sheets shown in Fig. S5 & S6. See §7.

Fig. S1: Experimental schematic. Red arrows: beam path from the probe laser (Laser 1). Blue Arrows: beam path from control laser (Laser 2). Purple arrows: overlapped beam path of the two lasers. Black arrows: electronic lines. Gray region: cryostat containing the optical cavity and membrane. The various components are described in §2.
Fig. S2: Schematic of the optical spectrum of laser tones. Red lines indicate tones produced from Laser 1. Blue lines indicate tones produced from Laser 2. The tones and their generation are described in §2. The gray curves indicate the two cavity modes used in this work.
**Fig. S3: Measurement of the dynamical back-action in the mechanical modes.** Here the cavity is driven with a single control tone, whose detuning is $\Delta$. Each panel shows the measured deviation of the mechanical mode’s eigenvalue from its bare value (i.e., from the relevant component of $\tilde{\lambda}^{(0)}$, whose numerical value is written in the panel). A global fit to standard optomechanical theory is used to extract the bare resonance frequencies $\tilde{\omega}^{(0)}$ and the optomechanical couplings $g$. See §3.
Fig. S4: Preliminary search for EP3. The quantity \( d(\Psi) \) (which ideally vanishes at \( \Psi_{\text{EP3}}^{(\text{thy})} \)), measured by scanning each of the four control parameters (one in each panel of the figure) through the theoretically estimated location of the EP3 point \( \Psi_{\text{EP3}}^{(\text{est})} = (2\pi \times 49.7 \text{ kHz}, 125 \mu W, 435 \mu W, 300 \mu W) \), which is used to select the six 2D slices shown in Fig. S5 and S6. The red curve is \( d(\Psi) \) calculated using the optomechanical parameters determined from fitting the knot (as described in §11). See §7.
**Fig. S5: Locating EP$_3$.** The quantity $d(\Psi)$ (which ideally vanishes at $\Psi_{EP3}$), measured on six 2D sheets passing through $\Psi_{EP3}^{(est)}$, the location of the EP$_3$ that is estimated from the 1D scans shown in Fig. S4. Top row: raw data. Middle row: data after outlier rejection and smoothing described in §9. The black circles show the minima that are located using the algorithm described in §9. Bottom row: the values of $d$ calculated from the optomechanical model.
**Fig. S6: Locating EP₃.** The data sheets of Fig. S5 arranged in 3D to illustrate the minimum of $d(\Psi)$ in the neighborhood of the experimental estimate of the location of the EP₃ ($\Psi_{EP₃}^{(est)}$). See §7.
Fig. S7: The locations of the sixty-one 2D sheets within $\mathcal{S}$. The sheets are color-coded by the 3D face in which they lie. (A) The sheets are shown within each of the eight 3D faces of $\mathcal{S}$. (B) The same sheets as in (A), shown using the “rectilinear stereographic” projection of Fig. 4B. Note that in this projection, all of the sheets are contained within the plot’s bounding box. (C) The same sheets, shown using the stereographic projection of Fig. 4A. The thin black lines show the boundary of each sheet. Thin gray lines show where a sheet exits the plot’s bounding box. The sheets are described in §8, and the projections are described in §10.
Fig. S8: The knot of EP$_2$ via four different signatures. The same data as in Fig. 4 of the main text, but in separate plots for the EP$_2$ locations determined by each of the four different signatures. (A) Zeroes of the discriminant $D$. (B) Phase vortices of the discriminant $D$. (C) Zeroes of the eigenvector indicator $E$. (D) Phase vortices of the eigenvector indicator $E$. The quantities $D$ and $E$ are defined in the main text. The projections used here are the same as in Fig. 4 of the main text. The solid curve is the same in all eight panels, and is the best-fit knot shown in Fig. 4 of the main text.
Fig. S9: **Control loops for the braids shown in the main text.** The three control loops in Fig. 5A-C of the main text were assembled from data taken in the two 2D sheets shown here. The two sheets’ common border is shown as the dashed gray line. Each of the small gray circles represents a value of $\Psi$ at which $\lambda$ is measured (i.e., a “pixel” in the 2D sheets of Fig. S13). The black crosses show the location of the EP$_2$ in these sheets as determined by the minima-finding algorithm described in §9.3.
Fig. S10: Additional braids of eigenvalues. As in Fig. 5 of the main text, panels A-C show three closed paths (green, red, blue), each from a different homotopy class. They share a common base point (black sphere) and are non-self-intersecting. The measured knot $\mathcal{K}$ (yellow circles) and the best-fit knot (orange curve) are shown for reference. The projection used here is the same as in Fig. 4A of the main text. Panels D-F show the eigenvalue spectrum $\lambda(\Psi)$ as $\Psi$ is varied around a loop. The variable $\xi$ indexes the values of $\Psi$ (along each loop) at which $\lambda$ is measured. The black crosses show $\lambda$ at the start and stop of the loop. The dashed lines are guides to the eye.
Fig. S11: Comparison of measured and calculated braids. (A) – (F) These are the same panels as Fig. 5A-F of the main paper. They show the control loops (green, red, and blue in (A)-(C)) in relation to the measured knot (yellow circles) and the best-fit knot (orange curve). The resulting eigenvalue braids are shown in (D)-(F). (G) – (I) The eigenvalue spectrum as calculated using the optomechanical parameters determined from fitting the knot of EP2. The dashed lines are guides to the eye.
**Fig. S12: Noncommutation of control loops.** The eigenvalue braids generated by concatenating two control loops. (A) Two loops (red, blue) belonging to different homotopy classes. They are non-intersecting, except that they share a common base point (black sphere). The measured knot (yellow circles) and the best-fit knot (orange curve) are shown for reference. The projection used here is the same as in Fig. 4A of the main text. (B) The eigenvalue spectrum $\lambda(\Psi)$ as $\Psi$ is varied around the loop formed by concatenating the two loops in (A). Specifically, the red loop is traversed first ($1 \leq \xi \leq 59$), and then the blue loop ($60 \leq \xi \leq 116$). The black crosses show $\lambda$ at the start and stop of the loop. The dashed lines are guides to the eye. (C) The eigenvalue spectrum as $\Psi$ is varied first around the blue loop ($1 \leq \xi \leq 57$), and then the red loop ($58 \leq \xi \leq 116$). The non-commutation of the two control loops is evident from the fact that the braids in (B) and (C) are inequivalent.
Fig. S13: The sixty-one 2D data sheets used to locate the $Ψ_{EP2}$ in the hypersurface $𝓢$. Each panel shows the complex-valued quantities $D$ and $E$ measured on a 2D sheet in $𝓢$. From left to right, the columns show $\text{Abs}(D)$, $\text{Arg}(D)$, $\text{Abs}(E)$, and $\text{Arg}(E)$. In each sheet, two of the control parameters are held fixed (these fixed values are given in the upper right corner). The other two control parameters are scanned, and form the horizontal and vertical axes of the 12 panels. The top row shows the raw data. The middle row shows the data after outlier rejection and convolution with a Gaussian (described in §9). The cyan circles are the $Ψ_{EP2}$, which are identified algorithmically (also described in §9). The bottom row shows $D$ and $E$ as calculated from optomechanics theory. The cyan squares are the $Ψ_{EP2}$ determined from this calculation. In addition, each panel includes two views of the $Ψ_{EP2}$ data and best-fit knot (as in Fig. 4 of the main text) in which the specific $Ψ_{EP2}$ found in that panel are shown in red.
Figure S13 (cont’d)
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