Abstract

Nonreciprocal dynamics in a cryogenic optomechanical system

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Nonreciprocity in various branches of physics has been studied for more than a century, e.g., from classical to quantum mechanics, and from particle physics to condensed matter physics. It is particularly interesting to consider nonreciprocal phenomenon in open (non-hermitian) systems. In this dissertation, I use a cryogenic cavity optomechanical system to demonstrate robust nonreciprocal interactions between two phononic resonators. The nonreciprocity, either transient or static, is realized via the cavity mediated optomechanical interaction.

I will start with a pedagogical introduction to nonreciprocity as well as non-hermiticity, followed by a brief review of optomechanics and a theoretical derivation of nonreciprocity in optomechanical systems. Then I will introduce our experimental realization of an optomechanical system, i.e., the membrane-in-the-middle setup. Next I will present the main result of this dissertation, which includes the experimental demonstration of both transient and static optomechanical nonreciprocity. I will conclude with a discussion of the first steps in our on-going study of higher-order degeneracies in multi-level open systems (also known as exceptional points).

Nonreciprocal dynamics in a cryogenic optomechanical

system

A Dissertation Presented to the Faculty of the Graduate School of Yale University in Candidacy for the Degree of Doctor of Philosophy

> by Luyao Jiang

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List of Acronyms

- ADC Analog-to-digital converter point
- AM Amplitude modulation
- AOM Acousto-optic modulator
- AWG Arbitrary waveform generator
- **DP** Diabolic point
- EOM Electro-optic modulator
- **EP** Exceptional point
- FM Frequency modulation
- **FSR** Free spectral range
- LO Local oscillator
- MIM Membrane in the middel
- NLM Nonlinear medium
- **OFHC** Oxygen-free high-conductivity
- **OMIT** Optomechanically induced transparancy
- **OMIA** Optomechanically induced amplification
- PDH Pound-Drever-Hall
- **PSD** Power spectral density

- RF Radio frequency
- **RHS** Right hand side
- SNR Signal-to-noise ratio
- TTL Transister-transister logic
- **VCO** Voltage controller oscillator
- **VEP** Virtual exceptional point
- **ZPF** Zero-point fluctuations



Introduction to nonreciprocity and non-hermiticity

Reciprocity is a fundamental feature of a linear, time-invariant system. Etymologically, the adjective word "reciprocal" comes from the Latin word "reciprocus", which is possibly from a phrase such as *reque proque*, based on the prefix "re-" (back) and "prō" (forward) [1]. Therefore, being reciprocal essentially means "going the same way backward as forward". Physically, a reciprocal system exhibits a symmetry when its source(s) and detector(s) are interchanged. For example, if acoustic or electromagnetic waves can make their way from a source to a detector, the propagation in the opposite path (from the detector to the source) will have equal transmission.

Though reciprocity is important for the functionality and analysis of various physical systems, it can be advantageous to break it in some practical situations. Electrical diodes, for example, are nonlinear nonreciprocal devices that are of fundamental importance in electronics. The first linear passive nonreciprocal device called gyrator was proposed in 1948 [2], primarily to remove the need of bulky and expensive inductors in telephony systems. To date, various nonreciprocal devices such as isolators, circulators and directional amplifiers have been widely used, for example, to prevent destabilizing reflections entering sensitive sources, or to mitigate multi-path interference in a communication system [3–5].

A necessary condition for nonreciprocity is time-reversal symmetry breaking. In general, timereversal symmetric systems¹ are guaranteed to be reciprocal, while reciprocity itself does not require

^{1.} Time-reversal is represented by the operator \mathcal{T} such that $\mathcal{T}(t) \equiv -t$, and a system described by a state vector $\psi(t)$ is time-reversal symmetric if $\mathcal{T}{\psi(t)} = \psi(t)$.

time-reversal symmetry. Consider a system with absorption loss. The system is time-reversal asymmetric since reversed output on its way back through the system will be absorbed a second time and will never result in the initial input. However, this system is reciprocal since the transmission coefficients for the forward and backward propagations are equal.

Systems with loss and/or gain have no time-reversal symmetry, which makes them good candidates for inducing nonreciprocity. The evolution of such a system is often described by a nonsymmetric (more generally, non-Hermitian) dynamical matrix, which will be discussed later in this chapter.

This chapter is organized as follows. In Sec. 1.1, I will review reciprocity theorems, and present a proof of Lorentz reciprocity. Then in Sec. 1.2, I will describe several nonreciprocal devices and discuss ways to achieve nonreciprocity. In Sec. 1.3, I will introduce general non-hermiticity in physics, as well as the effective Hamiltonian and dynamical matrix for specific systems, and then focus on a particular degeneracy in open systems called the exceptional point (EP). The nonreciprocal behavior related to EP will be covered in Ch. 2 and Ch. 3.

1.1 Reciprocity theorems

Perhaps the simplest statement of a theorem on reciprocity would be "if I can see you, then you can see me." Historically, the first explicit description of a reciprocity theorem traces back to Stokes [6] and Helmholtz [7], even before the electromagnetic nature of light became known. The so-called Helmholtz reciprocity principle states that a ray of light and its reverse ray encounter identical optical adventures such as reflections, refractions, and absorptions in a passive medium or at an interface. This concept was later reformulated by Kirchhoff [8], and described by Rayleigh [9] as a consequence of the linearity in the propagation of small amplitude vibrations (e.g., of sound or light).

A general form of reciprocity theorem in classical electromagnetism, Lorentz reciprocity, is named after work by Hendrik Lorentz in 1896 [10]. Loosely speaking, it states that the relationship between a time-harmonic electric current and the resulting electromagnetic field is unchanged if one interchanges the points where the current is placed and where the field is measured. For the specific case of an electrical network, the theorem is often postulated as a statement that the current

at position A due to a voltage at B is identical to the current at B due to the same voltage at A.

Later on in the 1930s, Onsager derived the reciprocal relations bearing his name [11,12] for linear processes, based on the assumption of microscopic reversibility. These relations establish the equality of certain ratios between flows and forces in thermodynamic systems, and can be generalized to a variety of physical processes [13, 14] (e.g., transport of heat, electricity, and matter) and to nonlinear systems [15]. Specifically, the application of Onsager relations in the context of electromagnetic constitutive relations of linear, homogeneous materials yields the result of Lorentz reciprocity [16].

We now review a proof of Lorentz reciprocity theorem based on Maxwell's equations and vector operations [17]. Consider a volume V bounded by the surface S that contains two sets of sources J_1 and J_2 . For the sake of simplicity we assume that source J_i produces time-harmonic electric (magnetic) field E_i (H_i) at frequency ω_i , and that $\omega_1 = \omega_2 = \omega$. According to Maxwell's curl equations, for i = 1, 2 we have:

$$\nabla \times \boldsymbol{E}_i = -i\omega\boldsymbol{\mu}\boldsymbol{H}_i \tag{1.1}$$

$$\nabla \times \boldsymbol{H}_i = i\omega\boldsymbol{\varepsilon}\boldsymbol{E}_i + \boldsymbol{J}_i \tag{1.2}$$

where ε and μ denote permittivity and permeability, respectively. Now consider the quantity $\nabla \cdot (E_1 \times H_2 - E_2 \times H_1)$, which can be expanded using a vector identity as:

$$(\nabla \times \boldsymbol{E}_1) \cdot \boldsymbol{H}_2 - (\nabla \times \boldsymbol{H}_2) \cdot \boldsymbol{E}_1 - (\nabla \times \boldsymbol{E}_2) \cdot \boldsymbol{H}_1 + (\nabla \times \boldsymbol{H}_1) \cdot \boldsymbol{E}_2$$
(1.3)

With proper substitutions we can derive

$$\nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) = i\omega(\boldsymbol{E}_2 \boldsymbol{\varepsilon} \boldsymbol{E}_1 - \boldsymbol{E}_1 \boldsymbol{\varepsilon} \boldsymbol{E}_2 - \boldsymbol{H}_2 \boldsymbol{\mu} \boldsymbol{H}_1 - \boldsymbol{H}_1 \boldsymbol{\mu} \boldsymbol{H}_2)$$

$$+ \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1$$
(1.4)

The first term in the right-hand side (RHS) of Eq. (1.4) adds up to zero if ε and μ are scalars or symmetric tensors, yielding the Lorentz reciprocity relation (in differential form):

$$\nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) = \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1 \tag{1.5}$$

With the divergence theorem, the integral form of Eq. (1.5) can be written as:

$$\iiint (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot \boldsymbol{n} dS = \iiint (\boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1) dV$$
(1.6)

where n is the outward pointing unit normal field of the boundary S.

The above reciprocity theorem can be simplified when $J_{1,2}$ are localized such that the surface integral of Eq. (1.6) cancels. In this case we obtain the so called Rayleigh-Carson reciprocity theorem:

$$\iiint J_1 \cdot E_2 dV = \iiint J_2 \cdot E_1 dV \tag{1.7}$$

Note that $J_1 = J_2$ leads to $E_1 = E_2$ as a result from the arbitrariness in the choice of V, indicating the measurement of the field is insensitive to the interchange of source and detector locations. In antenna theory, the Lorentz reciprocity theorem ensures a symmetric impedance matrix for electrical networks. More generally, as pointed out in Ref. [18], if we associate the input and output of a reciprocal system with a scattering matrix S, where the elements of S are defined as $S_{ij} = b_i/a_j$ with a_i and b_j being the amplitudes of the signals at ports *i* and *j*, respectively, then $S^T = S$. For a nonreciprocal system, such symmetry of the scattering matrix is broken.

Our proof of Eq. (1.5) also gives hints on how to break Lorentz reciprocity, i.e., to make the first RHS term of Eq. (1.4) nonzero. Firstly, if the permittivity (permeability) tensor is asymmetric, the order in which E_1 , E_2 and ε (H_1 , H_2 and μ) are multiplied in Eq. (1.4) becomes important, and may lead to nonzero sum. The second scenario for nonreciprocity is when ε (μ) is a function of the electric (magnetic) field strength or direction. Finally, the system will be nonreciprocal in general if ε and μ are time-dependent. We will see in the next section that these scenarios correspond to three ways of breaking reciprocity, namely adding magnetic bias, introducing nonlinearity and applying temporal modulation.

1.2 Breaking reciprocity

The study of nonreciprocity in physics probably began with the experimental discovery by Faraday in 1845 that the light passing through glass in the direction of an applied magnetic field is subject to rotation of the plane of polarization, which is linearly proportional to the component of the magnetic

field in the direction of propagation². This so called the Faraday effect breaks time-reversal symmetry locally (that is, only the propagation of light but not the source of the magnetic field is considered) as well as Lorentz reciprocity, and plays an important role in modern commercial nonreciprocal devices [19–28].

Faraday effect in magneto-optical materials (i.e., a medium through which left- and right-handed elliptically polarized lights can propagate at different speeds by the presence of a quasi-static magnetic field) breaks reciprocity by introducing a magnetic bias. Biasing other quantities (e.g., direct electric current, linear or angular momentum) that break time-reversal symmetry may also induce nonreciprocity [29, 30]. Further implementations of nonreciprocal devices are based on, for example, acoustic or optical nonlinearities [31–39], stimulated Brillouin scattering [40–43], spatial-temporal modulation [4, 44–52], chirally coupled single atom [53, 54]. Note these implementations are not mutually exclusive, and a cited example may belong to more than one category.



Figure 1.1: Gyrator schematic adapted from Wikipedia. The arrow indicates the direction of gyration resistence.

Yet before turning into different ways of breaking reciprocity, we will briefly discuss idealized models of nonreciprocal devices, namely gyrators, isolators and circulators. An ideal gyrator is a lossless linear two-port device³ that couples the current on one port to the voltage on the other and vice versa. With unit gyration resistance, the instantaneous currents and voltages are related by $v_2 = i_1$ and $v_1 = -i_2$, so the scattering matrix is:

^{2.} It was unclear though, whether Faraday himself, being excited by the success in "magnetising a ray of light", noticed the nonreciprocity in this phenomenon.

^{3.} Two-port means two pairs of terminals connected to the ends of the device, with a pair of input and output terminal on each end.

$$S = \left(\begin{array}{cc} 0 & -1\\ 1 & 0 \end{array}\right) \tag{1.8}$$

Another two port device called the isolator transmits power in only one direction, while absorbing all the power entered from the other. The scattering matrix of an isolator is:

$$S = \left(\begin{array}{cc} 0 & 0\\ 1 & 0 \end{array}\right) \tag{1.9}$$

The third nonreciprocal device we want to mention is the circulator. At microwave frequencies, circulators are important for radars and for the design of full-duplex communication systems, which allow transmitting and receiving through the same frequency channel at the same time, offering the opportunity to increase channel capacity and reduce power consumption [55]. At optical frequencies, circulators are used not only in communication systems but also in sensing and imaging fields, because of their low insertion loss and high isolation between the input and output signals [56]. For an ideal three-port circulator, signals applied to port 1 only come out of port 2, signals applied to port 2 only come out of port 3, and signals applied to port 3 only come out of port 1. Thus its scattering matrix is:

$$S = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \tag{1.10}$$

We review three methods of inducing nonreciprocity in the remainder of this section.

1.2.1 Magnetic bias

Magnetic materials induce nonreciprocal responses based on the energy splitting of quantum states with opposite angular momentum in the presence of a static magnetic field [21]. Up to now, commercial nonreciprocal devices are mostly based on ferrites [57] (e.g., yttrium iron garnet and materials composed of iron oxides and other elements such as Al, Co, Mn, Ni). In general, a ferrite's nonreciprocity results from electron-spin precession described by the Landau-Lifshitz-Gilbert equation [58]:

$$\dot{m}(t) = -\gamma m \times H_{\text{eff}} - \lambda m \times (m \times H_{\text{eff}})$$
(1.11)

where m is the magnetic dipole moment, γ is the electron gyromagnetic ratio and λ is a phenomenological damping parameter. The effective field H_{eff} is a combination of the external magnetic field B_0 , the demagnetizing field (magnetic field due to the magnetization), and some other effects such as exchange interaction, crystalline anisotropy, magnetostatic self-energy, thermal fluctuations, etc. It is shown in Ref. [17] that the permeability μ of a ferrite is antisymmetric, so Eq. (1.5) does not hold.

To further understand how magnetic bias induces nonreciprocity, we look at the Faraday effect. When electromagnetic waves propagate through a magneto-optical material subject to a magnetic field, the plane of the polarization rotates clockwise or counterclockwise from the point of view of the observer. The medium is therefore dextrorotatory (associated with clockwise rotation) or levorotatory (associated with counterclockwise rotation) depending on the direction of propagation relative to the applied magnetic field. We can see from Fig. 1.2 that the rotation is antisymmetric for opposite directions of propagation, leading to nonreciprocal transmission.



Figure 1.2: Faraday effect for waves propagating in two (left and right) directions. The transmission is nonreciprocal for an unaltered external magnetic field.

In practice, the magnetic bias is usually provided by a permanent magnet, or a resistive/superconductive coil. Therefore ferrite-based devices tends to be bulky, heavy and costly. Meanwhile, the bandwidth of such devices is limited by the tuning rate of the external magnetic field. Ferrite-based devices are also not amenable to integrated circuit technology, because the ferrite crystal lattices are incompatible with those of semiconductor materials. These issues have generated interest in "magnetless" nonreciprocity.

1.2.2 Nonlinearity

A nonlinear medium (NLM) may break the Lorentz reciprocity as its permittivity depends on the external electric field. For such a medium, the induced polarization field can be expressed as a function of the applied electric field:

$$P = \varepsilon_0 \chi^{(1)} E + \varepsilon_0 \chi_1^{(2)} E^2 + \varepsilon_0 \chi^{(3)} E^3 + \dots$$
(1.12)

where ε_0 is the vacuum permittivity and $\chi^{(n)}$ is the *n*-th order component of the electric susceptibility of the medium. For simplicity, we have assumed the medium to be isotropic and treated the fields *P* and *E* as scalars in Eq. (1.12)⁴. The second and third order susceptibilities $\chi^{(2)}$ and $\chi^{(3)}$ are significant for nonlinear phenomena (e.g., sum-frequency generation, Kerr effect), and offer a variety of possibilities for realizing nonreciprocal transmission.



Figure 1.3: Nonreciprocity based on nonliearity. a, Schematic of an acoustic diode adapted from Ref. [32]. b, SEM picture of the core device in Ref. [38].

To demonstrate $\chi^{(2)}$ associated nonreciprocity, consider second-harmonic generation (SHG) in a NLM. SHG occurs efficiently when the incident wave is of high intensity, whereas the efficiency of the inverse process (i.e., spontaneous parametric down-conversion), approaches zero in the classical limit [60]. Nonreciprocal transmission can be induced by combining a SHG medium with either a frequency selective mirror or a linear attenuator⁵. Fig. 1.3a shows the design in Ref. [32], where a superlattice is coupled with a NLM. When an acoustic wave with fundamental frequency is incident

^{4.} We have also assumed that the polarization at time t depends only on the instantaneous electric field. According to Kramers–Kronig relations, such instantaneous respondence implies that the medium is dispersionless and lossless. General expression for the susceptibility of an anisotropic NLM can be found in Ref. [59].

^{5.} More generally, nonreciprocal transmission could be observed in a sequence of two devices with power-dependent input–output functions g and h, providing $g(h(x)) \neq h(g(x))$.

from the left, it is strongly reflected by the superlattice. When the same signal comes from the right, the NLM generates some second harmonic wave that can be transmitted through the superlattice, thereby breaking the reciprocity. Note that while the transmission at the fundamental frequency is small in both directions, frequency conversion is largely asymmetric so the total power is transmitted from only one side.

Another example of $\chi^{(2)}$ -based nonreciprocity is demonstrated in Ref. [38] and is shown in Fig. 1.3b. Three optical modes coexist in the microring resonator, and are coupled by $\chi^{(2)}$ nonlinear interaction. A Hamiltonian $H_{int} = g(\hat{a}^{\dagger}\hat{b}\hat{c}^{\dagger} + \hat{a}\hat{b}^{\dagger}\hat{c})$ captures such a three wave mixing process, and the momentum conservation law requires counterclockwise (CCW) light in mode *a* can only interact with CCW light in modes *b* and *c*. Applying a CCW drive in mode *a* induces nonreciprocal transmission for modes *b* and *c*, since the CCW modes *b* and *c* interact with the drive, while the corresponding clockwise (CW) modes remain decoupled.

Nonreciprocity has also been explored with third order ($\chi^{(3)}$, or Kerr) nonlinearity [34,61–66]. For instance, if a Kerr material is engineered to have different field distribution when excited from opposite directions, the transmitted signal will be asymmetric, and very large isolation can be achieved with appropriate design [34,63].

There are other efforts to exploit nonlinear nonreciprocity. Static nonreciprocity is realized in mechanical metamaterials, by combining large nonlinearities with spatial asymmetries [67]. Similarly, coupling acoustic metamaterial to a nonlinear electronic circuits yields nonreciprocal phonon transport [36]. Bifurcation and chaos as the source of nonlinear frequency conversion also offer an avenue for acoustic rectification [33].

Nonlinearity-based nonreciprocal devices are under intensive study, yet several undesirable features have prevented them from practical applications so far. Firstly, the response of such devices is inherently dependent on the input intensity, which makes the devices prone to distort the incident signal and only perform well at specific levels of input power. Secondly, many nonlinear isolators are subject to a fundamental trade-off between the transmission coefficient in the forward direction and the nonreciprocal intensity range [66]. Thirdly, some nonlinear systems are shown to exhibit "dynamic reciprocity" [64] when a small-amplitude input (e.g., noise) is superimposed on a large-amplitude input (e.g., signal), such that the transmissions of the small-amplitude input are the same for both forward and backward directions.

1.2.3 Temporal modulation

Temporal modulation may induce nonreciprocity by breaking the time invariance of a system. This idea was recognized in microwave and optical systems nearly six decades ago [44, 45], and has got broad attention recently, thanks to the technological advances in realizing efficient time-modulated systems [4, 68].

We first illustrate nonreciprocity based on traveling wave modulation, proposed in Ref. [46] and implemented in Ref. [48]. Consider a waveguide that supports two orthogonal modes at frequencies ω_1 and ω_2 , with a modulated electric permittivity:

$$\varepsilon(x, y, z, t) = \varepsilon_{st} + \Delta \varepsilon(x, y) \cos(\omega_m t - k_m z)$$
(1.13)

where ε_{st} is the static permittivity⁶, $\Delta \varepsilon(x, y)$ is the modulation profile across the waveguide crosssection, $\omega_m = \omega_1 - \omega_2$ is the modulation frequency, k_m is the modulation wavenumber and z is the coordinate parallel to the waveguide axis. We notice the time-reversal symmetry is broken by the directionality of the modulation Eq. (1.13). Meanwhile, the two waveguide modes with frequencies and wavevectors $(\omega_1, k_1 \hat{z}), (\omega_2, k_2 \hat{z})$ are coupled if the modulation scheme satisfies the phase-matching condition (i.e., $k_m = k_2 - k_1$). Under this condition, the modulation can scatter any of the two modes (grey lines in Fig. 1.4a) to the other one, leading to complete mode conversion over the coherence distance (a quantity that is inversely proportional to the overlap integral between $\Delta \varepsilon(x, y)$ and the mode profiles over the waveguide's cross-section). Such modulation-mediated mode conversion process is only possible for one propagation direction, since in the opposite direction a different phase-matching condition (i.e., $k_m = -k_2 + k_1$) is not satisfied. Therefore, the transmission through such a traveling wave modulated waveguide is nonreciprocal.

Instead of the directional coupling between co-propagating waves as described above, modulation can also couple modes that propagate in opposite directions. The effect analogous to the Doppler effect observable in a mechanically moving Bragg grating [69, 70] has been used to implement nonreciprocal transmission in materials biased by two counter-propagating optical signals with slightly detuned frequencies [71, 72].

^{6.} The permittivity here is written as a scalar rather than a tensor for simiplicity.



Figure 1.4: Nonreciprocity based on temporal modulation. a, Schematic of raveling wave modulation adapted from [4]. The modulation creates an indirect transition between two waveguide modes. b, Illustration of the Aharonov-Bohm effect and a designed optical isolator based on a pair of modulated waveguides, adapted from [46]. $|1\rangle$ and $|2\rangle$ can represent either distinct quantum states of electrons, or two orthogonal modes in the waveguides.

The nonreciprocity realized in traveling wave-modulated materials (described in the previous paragraph) is a result of indirect transitions, where both frequency and momentum of signals are varied. We have seen that such transition is inherently unidirectional. Direct transitions, in contrast, only affect the frequency of the signals and have no inherent unidirectionality. A single direct transition cannot break reciprocity; however by combining two or more such transitions, one may induce nonreciprocal transmission. This is because the modulation provides an effective gauge field, and an interference phenomenon analogous to the Aharonov-Bohm effect (upper Fig. 1.4b) can be used to realize nonreciprocal devices [73–75].

As an example, an isolator can be realized using based on two modulated waveguides and one waveguide without modulation [73]. The waveguides support two orthogonal modes *a*, *b* and are uniformly modulated along *z* axis with a frequency equal to the difference of the mode frequencies. The modulation covers the upper half of the waveguides (black region in lower Fig. 1.4b), converts one mode to the other and causes an accumulation in phase of the converted signal. When a signal (say, in mode *a*) propagates through this structure, it can either stay unconverted (in mode *a*) or be converted twice by the modulation (via $a \rightarrow b \rightarrow a$), and the transmitted signal (in mode *a*) is the interfered result between these two paths. Suppose the waveguides are modulated with different phases, the total phase accumulated in the converted signal is $\phi_a^{L\rightarrow R} = \phi(z_1) + \phi_{s2} - \phi(z_2)$ for transmission from left to right, where $\phi(z_1)$ and $\phi(z_2)$ are the modulation phase of the left and right waveguide, respectively, and ϕ_{s2} is the reciprocal propagation phase of mode *b* through the unmodulated center waveguide. For transmission from right to left, we have $\phi_a^{R\rightarrow L} = \phi(z_2) + \phi_{s2} - \phi(z_1)$.

On the other hand, the unconverted signal accumulates the same phase ϕ_{s1} in both directions. Since the structure converts only half of the input power, if we select $\phi(z_1) - \phi(z_2) = \phi_{s1} - \phi_{s2} = \pi/2$, the converted and non-converted parts of the incident signal interfere destructively (constructively) at the right (left) end of the structure, leading to zero (unitary) transmission at the corresponding end, and thus making the structure operate as an isolator.

Temporal modulation offers an interesting opportunity towards compact, integrated nonreciprocal devices, yet it comes with some challenges. Devices based on traveling wave modulation are subject to a trade-off between device length, modulation power and bandwidth. At optical frequencies, (magnetless) time modulated devices are limited by the ability to induce strong, fast and robust modulation, therefore remain at a proof-of-concept level, compared with their counterpart at microwave frequencies [4].

In the end of this section, we emphasize that the nonreciprocity induced from multi-path interference in a coupled-modes system is an essential idea throughout this dissertation. As we will see in next chapter, both optical and mechanical modes in an optomechanical system can easily be parametrically modulated in time, which enables us to induce nonreciprocity using concepts similar to those presented here.

1.3 Non-Hermitian physics

Hermitian operators play an important role in quantum mechanics, as they yield real eigenvalues that usually represent the measurement of corresponding physical quantities⁷. In particular, the Hamiltonian operator, which describes the energy of a quantum system, is Hermitian when the system is closed. However, if someone merely cares about a subsystem where energy can be transferred to its environment, hermiticity of the subsystem's effective Hamiltonian is not guaranteed. Complex energy eigenvalues were introduced to describe the tunneling rate of a particle in a pioneering study on alpha decay by George Gamow [77], where the real (imaginary) parts of these eigenvalues correspond to the energy levels (resonance widths) observed in experiments. In a subsequent work [78], complex

^{7.} In Ref. [76], a class of non-conservative Hamiltonians that commutes with the parity-time operator was shown to yield entirely real spectra. This discovery indicates that observables in quantum mechanics may not necessarily be described by Hermitian operators, and has brought us upon a vast amount of study on PT symmetric systems. However such topic is beyond the scope of the dissertation.

potentials were used to describe the scattering interactions between neutrons and nuclei.

These early studies using on non-Hermitian quantum physics were mostly phenomenological, while more rigorous approaches [79, 80] were developed thereafter to describe the dynamics of open quantum systems. It turns out that when a quantum system couples to a surrounding environment, its dynamics becomes non-Hermitian and quantum jumps will occur [81]. A microscopic view of such an open system reveals the necessity to account for the noise induced by quantum jumps, in order to keep the quantum mechanical commutation rules intact [80]. Macroscopically, such noise is neglected in semiclassical approaches, and the quantum dissipation process can be encapsulated in a non-Hermitian "effective" Hamiltonian.

Traditionally, adding non-Hermitian components to a (Hermitian) Hamiltonian has been regarded as a perturbation, with the physics essentially same as the Hermitian case, except for an exponential decay. However in presence of degeneracies (i.e., coalescences of eigenvalues), non-Hermitian terms can do significantly more than broadening the resonances of the system and allowing its eigenstates to decay [82, 83].

In this section, I first describe a mapping between an *N*-level quantum system and *N* coupled classical oscillators, and discuss the non-Hermitian terms in both systems. Then I will examine the degeneracies involving two states (i.e., N = 2) for Hermitian and non-Hermitian matrices.

1.3.1 Effective Hamiltonian and dynamical matrix

Let us consider a first order linear ordinary differential equation:

$$\dot{\boldsymbol{a}}(t) = -i\boldsymbol{B}\boldsymbol{a}(t) \tag{1.14}$$

where *a* is an *N*-vector and *B* is an *N* × *N* matrix. Suppose the coefficient matrix *B* is constant and Hermitian, such that there exist *N* real eigenvalues λ_n and corresponding orthogonal eigenvectors v_n such that $Bv_n = \lambda_n v_n$. The solution to Eq. (1.14) is therefore:

$$\boldsymbol{a}(t) = \sum_{n=1}^{N} \boldsymbol{v}_n \left\langle \boldsymbol{v}_n, \boldsymbol{a}(0) \right\rangle e^{-i\lambda_n t}$$
(1.15)

where $\langle x, y \rangle$ represents the inner product of two vectors x and y.

In quantum mechanics (QM), B and a(t) can be interpreted as the Hamiltonian matrix and the state vector, respectively, so Eq. (1.14) is the Schrödinger equation of a (closed) *N*-level system. λ_n (v_n) represent the energy eigenvalues (eigenstates). If B is Hermitian and time dependent, then Eq. (1.14) (with $B \rightarrow B(t)$) still describes an *N*-level quantum system, now subject to the explicitly time dependent Hamiltonian B(t).

In classical mechanics (CM), Eq. (1.14) is the equation of motion of N coupled linear undamped oscillators. This can be seen, for example, by rewriting Eq. (1.14) as $\ddot{a}(t) + B^2 a(t) = 0$. However Eq. (1.14) remains to be a useful form for describing classical oscillators for a number of reasons. First, for a constant B, the eigenvalues and eigenvectors of B correspond to the normal modes and normal frequencies, while the complex vector a(t) encodes the displacements and momenta of individual oscillators (in its real and imaginary parts, separately). In this CM context, the matrix B is referred to as the dynamical matrix (although in some literature it is called the "Hamiltonian" in a mathematical analogy with the QM case, it is important to note that the B is not the Hamiltonian function H(q, p) of CM). Second, in a classical system with explicit time-dependence, the mathematical form of Eq. (1.14) (with $B \rightarrow B(t)$) allows all of the familiar results from QM to be applied to the corresponding features of the classical system. This includes such powerful results as the adiabatic theorem [84], Landau-Zener transitions [85, 86], and the geometric (Berry) phases [87–89].

From a mathematical point of view, we may consider Eq. (1.14) when the elements of matrix B are arbitrary complex numbers, making B not necessarily Hermitian. We reconsider the physical systems described by Eq. (1.14). Interpreted in the context of QM, i.e., for the *N*-level quantum system, a non-Hermitian B would correspond to an effective Hamiltonian, which may be used to describe a system that interacts with the environment and is subject to dissipation. For the *N*-oscillator classical interpretation, a non-Hermitian dynamical matrix means that we have included dampers and gyrators in the system⁸. More details of the isomorphism between the quantum description of N-level systems and the classical description of coupled oscillators can be found in Ref. [90–92].

^{8.} A damper adds a term proportional to velocity in the classical equation of motion, which can be achieved by, say, submerging a mechanical oscillator into a fluid. A gyrator creates nonreciprocal coupling between two 1D oscillators. One example for a mechanical gyrator is a charged ball oscillating in a 2D quadratic potential well $V(x, y) = x^2 + y^2$ with a magnetic field in *z* direction, where the oscillation in *x* and *y* degrees of freedom are coupled antisymmetrically.

1.3.2 Degeneracies of Hermitian matrices

General complex Hermitian matrices have degeneracies of codimension three, that is, degeneracies are isolated points in a three-parameter space [83]. Such degeneracies are robust against Hermitian perturbations, as they are simply moved but not destroyed when extra parameters are varied. A special kind of Hermitian matrices is real symmetric ones, which describe systems that are non-dissipative and time-reversal symmetric. The degeneracies of these matrices are of codimension two, being isolated points in a two-parameter space. Consider a traceless 2×2 real symmetric matrix⁹:

$$H = \begin{pmatrix} \Delta & g \\ g & -\Delta \end{pmatrix}$$
(1.16)

where g and Δ are two parameters to vary. The eigenvalues are $\lambda_{\pm} = \pm \sqrt{g^2 + \Delta^2}$, with a degeneracy located at the origin ($g = \Delta = 0$). One can prove that the two associated eigenvectors are always orthogonal to each other for $\lambda_+ \neq \lambda_-$, and for $\lambda_+ = \lambda_-$, one can still construct an orthogonal eigenvector basis. As shown in Fig. 1.5, in the three-dimensional space of (λ_{\pm}, Δ, g), the eigenvalues form conical surfaces with apex at the degeneracy, and therefore such a degeneracy is referred to as a "diabolical point" (DP) [93].



Figure 1.5: Eigenvalues of a Hermitian matrix. a, Diabolo (double cone) shape of spectrum, with a DP at the origin. b, Avoided crossing of eigenvalues near the DP with g = 1 as Δ varies.

^{9.} The matrix can be viewed as either the Hamiltonian of a coupled two-level quantum system, or the dynamical matrix of two coupled classical oscillators in a rotating frame.

For an arbitrary Hermitian matrix, if one starts from a point in the parameter space and tracks the deformation/evolution of all eigenvectors along a closed loop (either surrounding the degeneracy in the real symmetric case here, or near a point of degeneracy in the complex Hermitian case), one will find that each eigenvector returns to its original form, apart from a phase factor determined by the continuation rule [94, 95]. In quantum mechanics, this corresponds to the fact that for a closed (Hermitian) system prepared in a particular eigenstate, if the parameters of its Hamiltonian are varied sufficiently slowly, the system will remain in the corresponding (i.e., smoothly connected) instantaneous eigenstate, which is the well-known adiabatic theorem [96].

1.3.3 Degeneracies of non-Hermitian matrices

For complex non-Hermitian matrices, degeneracies are of codimension two regardless of their symmetry. At such degeneracies, not only the eigenvalues coalesce, but also the eigenvectors become parallel. As an example, consider a symmetric 2×2 non-Hermitian matrix:

$$H' = \begin{pmatrix} \Delta + i & g \\ g & -\Delta - i \end{pmatrix}$$
(1.17)

which in classical mechanics corresponds to adding unit loss and gain to the uncoupled two-mode closed system. The eigenvalues of H' are given by $\lambda'_{\pm} = \pm \sqrt{g^2 + (\Delta + i)^2}$, with double degeneracies in (Δ, g) space at $(0, \pm 1)$. We plotted the real and imaginary parts of the eigenvalues separately, in the vicinity of one degenerate point, as shown in Fig. 1.6. One can observe that the eigenvalues have a sharper dependence on parameters as they approach the degeneracy, compared with the Hermitian case.



Figure 1.6: Eigenvalues of a non-Hermitian matrix. The real part and imaginary part of the eigenvalues are plotted as functions of parameters (Δ, g) .

An even more intriguing feature near degeneracies of a non-Hermitian matrix is the non-trivial monodromy, that is, the eigenvalues and the corresponding eigenvectors interchange for a circuit encircling the degeneracy. Such degeneracies, often referred to as "exceptional points" (EPs), are not diabolical points but rather branch points associated with the Riemann sheets of the function $\lambda(\Delta, g)$. If the non-Hermitian matrix is viewed as a perturbation of a real symmetric matrix in a planar parameter space, such perturbation splits one DP into two EPs. If the non-Hermitian matrix is a perturbation of a complex Hermitian matrix, and the parameter space is three-dimensional, the isolated Hermitian degeneracy point may expand into a ring of exceptional points. We show this according to Ref. [83]. For a matrix

$$M = \begin{pmatrix} z_0 & x_0 + iy_0 \\ x_0 - iy_0 & -z_0 \end{pmatrix} + i \begin{pmatrix} z & x + iy \\ x - iy & -z \end{pmatrix}$$
(1.18)

where $\mathbf{r_0} = (x_0, y_0, z_0)$ is the three-dimensional parameter space of the Hermitian matrix and $\mathbf{r} = (x, y, z)$ is the parameter space of non-Hermitian perturbations, the eigenvalues are expressed as $\lambda_{\pm}^M = \pm \sqrt{r_0^2 - r^2 + 2i\mathbf{r} \cdot \mathbf{r_0}}$. Thus for given vector $\mathbf{r_0}$, an "exceptional ring" with radius $\mathbf{r} = \mathbf{r_0}$ in the plane perpendicular to $\mathbf{r_0}$ is formed. A recent experiment has demonstrated this feature in a Si₃N₄ photonic crystal slab [97].

1.3.4 Encircling an EP: nonreciprocal state transfer

We now consider a state vector $c(t) = (c_1(t), c_2(t))^T$ of a two-mode system, whose evolution is governed by Eq. (1.14) with **B** replaced by either *H* defined in Eq. (1.16) or *H'* defined in Eq. (1.17). For a given c(0) and a time interval [0, T], we are interested in c(T) whereas the parameters (Δ, g) trace out a loop during this time interval. A particular interesting case is when the parameter loop encloses the degeneracy of the system.

Suppose the system is in one eigenstate at t = 0. When the system is Hermitian (i.e., described by *H* in Eq. (1.16)), and both Δ and *g* vary slowly (i.e., $|\frac{d\Delta}{dt}|, |\frac{dg}{dt}| \ll |\lambda_+ - \lambda_-| = 2\sqrt{g^2 + \Delta^2}$), then according to the adiabatic theorem, the system remains in the same instantaneous eigenstate throughout [0, T]. For a closed-loop variation of the parameters, this indicates c(T) = c(0). When the system is non-Hermitian (i.e., described by *H'*), naive application of the adiabatic theorem would suggest that the monodromy of the eigenvalue sheet would cause the system to evolve to the other eigenstate, such that a state transfer occurs and $c(T) \neq c(0)^{10}$. However, we now show that whether such a state transfer can happen depends on the initial conditions of the system as well as on the sense (clockwise or counterclockwise) of the parameter loop. Furthermore, the state transfer is nonreciprocal for a certain loop sense.

^{10.} If *H*' is symmetric, a stronger orthogonality condition $\langle c(T), c(0) \rangle = 0$ applies.



Figure 1.7: Evolution of the weighted eigenvalue. The (Δ, g) loop is counterclockwise with $(r_0, \phi_0, T) = (0.5, -0.5, 10)$. a, b, The real part and the imaginary part of $\overline{\lambda}(t)$ (black lines), on top of the eigenvalue sheets of the system. At t = 0 the system is initialized on the orange manifold (with a state vector $\lambda'_{-}(0)$). At t = T the system has transferred to the blue sheet. c, d, The trajectory of $\overline{\lambda}(t)$ with the same loop but a different initialization. A non-adiabatic jump occurs during [0, T] and there is no state transfer at the end of the loop.

For the sake of simplicity, I assume the parameter loop to be a circle describe by:

$$\Delta(t) = \Delta_0 + r_0 \cos(\frac{t}{T} + \phi_0)$$
(1.19)

$$g(t) = g_0 + r_0 \sin\left(\frac{t}{T} + \phi_0\right) \tag{1.20}$$

where $(\Delta_0, g_0) = (0, 1)$ is the loop center at an EP, r_0 is the radius and ϕ_0 is a phase offset¹¹. For given c(0) and (r_0, ϕ_0, T) , the differential equation $\dot{c}(t) = -iH'(\Delta(t), g(t))c$ can be solved numerically.

At any time $t \in [0, T]$, the solution c(t) can be written as:

$$c(t) = c_{+}(t)v_{+}(t) + c_{-}(t)v_{-}(t)$$
(1.21)

where $v_{\pm}(t)$ are instantaneous normalized eigenvectors of $H'(t) = H'(\Delta(t), g(t))$ that correspond to eigenvalues $\lambda'_{\pm}(t)$. Note $v_{\pm}(t)^{T}v_{-}(t) = 0$ due to the symmetry of H'(t). To visualize the system's evolution, we introduce an instantaneous "weighted eigenvalue":

$$\bar{\lambda}(t) = \frac{|c_{+}(t)|^{2}\lambda_{+}'(t) + |c_{-}(t)|^{2}\lambda_{-}'(t)}{|c_{+}(t)|^{2} + |c_{-}(t)|^{2}}$$
(1.22)

which represents the extent to which the state vector is projected onto one of the two eigenvectors (e.g., $c(t) \parallel v_{\pm}(t) \Leftrightarrow \overline{\lambda}(t) = \lambda'_{\pm}(t)$). As the state vector evolves, we can show the real and imaginary traces of $\overline{\lambda}(t)$ in the eigenvalue sheet of the system.

An example is illustrated in Fig. 1.7, where a counterclockwise, "adiabatic"¹² parameter loop encircling an EP is performed. In Fig. 1.7a, b, the system is initialized with $v_{-}(0)$, and at time t = T the state is transferred to $v_{+}(0)^{13}$. In contrast, the system initialized with $v_{+}(0)$ ends up in the same eigenstate (i.e., $c(T) \parallel v_{+}(0)$) after the loop, as shown in Fig. 1.7c, d. This result shows that the state transfer is nonreciprocal.

If we describe the evolution of the system via a propagator matrix $U_{\cup}(t)$ (with \cup indicating the loop is counterclockwise) such that $c(t) = U_{\cup}(t)c(0)$, the state transfer is nonreciprocal when $U_{\cup}(T)$ is asymmetric. We calculate and plot the elements of $U_{\cup}(T)$ as a function of T in Fig. 1.8. One observation is that $U_{\cup}(T)$ becomes more asymmetric as T increases.

Similar nonreciprocal state transfer can be observed with clockwise parameter loops. The propagator matrix $U_{\cup}(T)$ turns out to be the transpose of $U_{\cup}(T)$, since the clockwise and counterclockwise loops are time-reversals of each other (see in App. A). In general, the state transfer in a

^{11.} Note here the loop is counterclockwise in (Δ, g) space. To describe a clockwise loop, just set $t \to -t$.

^{12.} Similar to the Hermitian case, we define "adiabatic" to mean $\left|\frac{d\Delta}{dt}\right|, \left|\frac{dg}{dt}\right| \ll |\lambda'_{+} - \lambda'_{-}| = 2\sqrt{g^{2} + (\Delta + i)^{2}}$.

^{13.} The final state vector c(T) is not necessarily equal to $v_+(0)$, as the system is physically up to gain and loss. However, we can always normalize c(T) to quantize the efficiency of such state transfers.

non-Hermitian system depends on the geometry as well as the (non)adiabaticity of parameter loops (see in Ref. [98–101] for more details).



Figure 1.8: Elements of the propagator matrix, for the loop shown in Fig. 1.7 with varying loop time *T*, calculated numerically with Mathematica. In the diabatic time regime $(T \ll 1/r_0 \sim 1)$, the propagator matrix is approximately the identity matrix. The propagator matrix becomes asymmetric $(U_{\cup,12}(T) \neq U_{\cup,21}(T))$ as the loop goes adiabatic $(T \gg 1)$, indicating the state transfer is now nonreciprocal.

1.3.5 Further remarks

In the past several years, there has been a growing interest in studying non-Hermitian systems, especially in the presence of EPs [81, 102, 103]. A summary of codimensions of DP and EP in 2×2 matrices is shown in Table. 1.1. It is also natural to consider degeneracies involving three or more states, where high order EPs may come into place [104, 105]. For example, the third order degeneracies have been investigated both theoretically [106–109] and experimentally [110, 111]. I will present more details on high order EPs in Ch. 6.

To experimentally study non-hermiticity in physics, systems involving lasers and cavities seem to be promising candidates, since the gains are generated via external pumping, while the losses are already in place due to the input–output coupling of the cavities and the dissipation inside. The optomechanical platform used in this dissertation will be discussed in next two chapters, and it will be seen that EPs (and the "adiabatic" transport around them), as well as static nonreciprocity can be demonstrated by using optomechanical interactions to induce a non-Hermitian dynamical matrix.

Matrix type	Codimension of DP	Codimension of EP
Real symmetric	2	non-existent
Real asymmetric	3	1
Hermitian	3	non-existent
Complex symmetric	4	2
Complex asymmetric	6	2

Table 1.1: Codimensions of eigenvalue degeneracies, based on results from the singularity theory [112].



Nonreciprocity in optomechanics

With general nonreciprocity and various nonreciprocal devices being introduced in the previous chapter, let us narrow down to the nonreciprocal behavior in optomechanical systems. In a typical cavity optomechanical system [113], radiation pressure couples light within the optical cavity to one or more mechanical resonators, and offers a versatile way of producing temporal modulations. The time modulated optomechanical coupling, which induces effective interactions between different photonic (phononic) modes, opens the possibility for nonreciprocal light (sound) transmission. From a practical point of view, the optomechanical approach to building nonreciprocal devices may have several advantages such as the potential for linear response, low noise level, optical real-time reconfigurability and small system size [114].

This chapter begins with a pedagogical introduction to the field of cavity optomechanics, along with a mathematical framework of our membrane-in-the-middle setup. Next in Sec. 2.2 I will address the theoretical description and experimental implementations of nonreciprocal photon transmission in optomechanical systems. In Sec. 2.3, I will describe nonreciprocal phonon transmission, with a recapture of some previous work in our group, as well as the motivation to the main result in this dissertation.
2.1 Brief introduction to cavity optomechanics

The field of optomechanics explores the interaction between electromagnetic radiation and mechanical motion via radiation pressure. Effects of such optomechanical interaction was first considered in interferometric gravitational wave detectors. Over the past few decades, a broad range of optomechanical systems have been implementated. These systems are completely different in size (from nanometers to kilometers), mass (from attograms to kilograms) and frequency (from hertz to gigahertz), yet they are almost equivalent in their theoretical description.

The rapidly growing interest in optomechanics is driven by several motivations. Fundamentally, optomechanics allows coherent quantum control over the motion massive mechanical objects and the generation of macroscopic superposition states. It paves a new way for testing the validity of quantum theory, such as decoherence, which describes the process of objects transitioning from states that are described by quantum mechanics to states that are described by Newtonian mechanics. Pragmatically, optomechanical devices may find applications in sensing (e.g., sensitive optical detection of small forces or displacements) and quantum information processing (e.g., interconversion between stored solid-state qubits in the microwave domain and flying photonic qubits in the near-infrared domain).

In cavity optomechanical systems, the radiation pressure interaction is enhanced as electromagnetic modes are confined in an optical resonator (cavity). Such a system is no more than a bunch of damped harmonic oscillators that are parametrically coupled with each other. Therefore I will first review the general solution to the equation of motion that describes damped harmonic oscillators, and then derive the input-output properties of an optical cavity, before I discuss the optomechanical interaction and the standard linearization procedure for it. I will then review the model of the membrane-in-the-middle setup, which provides the optomechanical platform for the work in this dissertation.

2.1.1 Damped harmonic oscillators

The second order ordinary differential equation (ODE) that describes the position x(t) of a driven damped harmonic oscillator is:

$$\ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = \frac{F(t)}{m}$$
(2.1)

where ω_0 is the oscillator's natural frequency, γ is the damping rate or linewidth, *m* is the mass and *F*(*t*) is the driving force. To solve this equation, we transform it into the Fourier domain:

$$\chi_0^{-1}(\omega) \cdot x[\omega] = \frac{F[\omega]}{m}$$
(2.2)

where the Fourier transform for a function f(t) is defined as $f[\omega] = \int dt e^{i\omega t} f(t)^1$, and we have introduced a susceptibility function

$$\chi_0(\omega) = \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \to \frac{1}{2i\omega_0} \cdot \frac{1}{\gamma/2 - i(\omega - \omega_0)} \quad \text{as } Q \equiv \omega_0/\gamma \to +\infty$$
(2.3)

which represents the oscillator's response to its driving force at certain frequency. It is worth pointing out that the susceptibilities of the mechanical and optical oscillators in this dissertation are in the high-Q limit.

This high-Q limit approximation of susceptibility actually corresponds to the rotating wave approximation (RWA). To see this, we first introduce the momentum $p(t) = m\dot{x}(t)$ and rewrite Eq. (2.1) as two coupled first order ODEs:

$$\dot{x}(t) = \frac{p(t)}{m} \tag{2.4}$$

$$\dot{p}(t) = -\gamma p(t) - m\omega_{\rm m}^2 x(t) + F(t)$$
(2.5)

Then we define a pair of conjugate complex amplitudes:

$$c = \frac{1}{\sqrt{2m\hbar\omega_0}}(m\omega_0 x + ip) \tag{2.6}$$

$$c^* = \frac{1}{\sqrt{2m\hbar\omega_0}}(m\omega_0 x - ip) \tag{2.7}$$

such that they satisfy

^{1.} We use square brackets to represent the Fourier component of a function. For an arbitrary function *F*, one can show that $F^*[\omega] = (F[-\omega])^*$.

$$\dot{c} = -i\omega_0 c - \frac{\gamma}{2}(c - c^*) + \frac{iF(t)}{\sqrt{2m\hbar\omega_0}}$$
(2.8)

$$\dot{c}^* = i\omega_0 c^* + \frac{\gamma}{2}(c - c^*) - \frac{iF(t)}{\sqrt{2m\hbar\omega_0}}$$
(2.9)

We emphasize that our treatment is classical although the reduced Planck constant appears in the definition of $c^{(*)}$. In fact, \hbar in $c^{(*)}$ is introduced merely for normalization purposes, and can be interpreted as the area element² in phase space. This normalization is a standard practice in the field, as it allows one to easily move into a quantum description of the oscillator. As a result of choosing this normalization, we have $|c^{(*)}|^2 = E/\hbar\omega_0$ which represents the number of phonons, and $c^{(*)}$ corresponds exactly to the annihilation (creation) operator $\hat{c}^{(\dagger)}$ when the position *x* and momentum *p* are replaced by quantum operators \hat{x} and \hat{p} .

Let $c_{in} = \frac{iF}{\sqrt{2m\hbar\omega_0}}$ and transform the above equation into the Fourier domain:

$$\left[\frac{\gamma}{2} - i(\omega - \omega_0)\right]c[\omega] \equiv \chi^{-1}(\omega)c[\omega] = \frac{\gamma}{2}c^* + c_{\rm in}[\omega]$$
(2.10)

where $\chi(\omega)$ denotes the susceptibility³. When the quality factor Q of the oscillator is high ($\omega_0 \gg \gamma$), the susceptibility is strongly peaked around the natural frequency ω_0 , and the c^* term on the RHS of Eq. (2.10) is akin to a far-off-resonant drive, which will be suppressed by the narrow linewidth (γ) and therefore can be ignored under the RWA. An equivalent way to understand this approximation is to consider Eq. (2.8) in a rotating frame where $\tilde{c}(t) = e^{-i\omega_0 t}c(t)$:

$$\dot{\tilde{c}}(t) = -\frac{\gamma}{2}\tilde{c}(t) - \frac{\gamma}{2}e^{-2i\omega_0 t}\tilde{c}^*(t) + e^{-i\omega_0 t}c_{\rm in}(t)$$
(2.11)

Since \tilde{c}^* term rotates many times during the mode's natural response time $\tau = 1/\gamma$, it has an average of zero and can be ignored. We assume the RWA holds throughout this dissertation, and that the

^{2.} A more accurate description of the area element is $h = 2\pi\hbar$.

^{3.} This definition differs from the susceptibility function introduced in Eq. (2.3), but is proportional to the high-Q limit of the previous definition. We will use this definition in the remainder of the dissertation.

complex amplitude is governed by the linear equation:

$$\dot{c}(t) = -(i\omega_0 + \frac{\gamma}{2})c(t) + c_{\rm in}(t) \iff c[\omega] = \chi(\omega)c_{\rm in}[\omega]$$
(2.12)

The discussion of damped harmonic oscillators ends with a microscopic view of the mechanism underlying damping. As we already know, for an undriven simple harmonic oscillator (i.e, $\gamma = 0$ and F(t) = 0), the equation of motion is time-reversal invariant, while for damped harmonic oscillators this symmetry is broken. To understand the origin of such irreversibility, we must realize that the system, in this case an oscillator, is damped via interactions with its environment (alternatively referred to as the thermal bath). The simplest thermal bath can be modeled by an infinite set⁴ of simple harmonic oscillators, each coupled linearly to the system [80]. It turns out that the system's interaction with the environmental modes will not only lead to damping of the system, but will also exert a stochastic driving force $\eta(t)$ on it⁵. In general, the relationship between fluctuation and damping is captured by the fluctuation-dissipation theorem. [119]

Some statistical properties of $\eta(t)$ can be derived by considering a collection of identical harmonic oscillators (known as a canonical ensemble). Firstly, since the ensemble average of the system's displacement is zero, we have $\langle \eta(t) \rangle = 0$ (where $\langle \rangle$ represents the ensemble average). Secondly, if Born-Markov approximation for the thermal bath is applicable (which means the thermal bath is always in thermal equilibrium and the memory time of the thermal bath is much shorter than the response time of the system), we may use Dirac delta function to describe the self-correlation of the stochastic drive:

$$\langle \eta(t)\eta(t')\rangle = D\delta(t-t') \tag{2.13}$$

where *D* is the diffusion parameter that scales with the thermal bath temperature *T*. Eq. (2.13) indicates that a classical harmonic oscillator coupled to a "cold" thermal bath (T = 0) will come

^{4.} The number of bath modes being infinity is necessary. Otherwise, according to Poincaré's recurrence theorem (or the quantum version proved in Ref. [115]), the system will eventually return to its initial state, meaning there is no damping during this period. A sufficient condition to obtain proper dissipation is the bath modes being continuously distributed in frequency. This can be appreciated by thinking of a waveguide (as the thermal bath) attached to a cavity (as the system). If the length of the waveguide is finite, it has an infinite but discrete spectrum. A cavity photon emitted into the waveguide will be reflected at the other end and be back into the cavity subsequently. However if the waveguide is infinitely long, its spectrum will be continuous, and any transmitted photon can radiate away such that the cavity is now dissipative.

^{5.} The rigorous treatment, i.e., quantum Langevin equations were first obtained in Ref. [116], and can now be found in many textbooks such as Ref. [117, 118]

to rest, subject to neither fluctuation nor damping (dissipation). In contrast, a quantum harmonic oscillator is driven by vacuum fluctuations even at zero temperature, which leads to phenomena such as spontaneous emission [120].

2.1.2 Input-output of optical cavities

Optical cavities (resonators) confine certain electromagnetic waves in space by allowing them to circulate in a closed path. Traditionally, an optical cavity consists of two highly-reflective mirrors, and the light in between bounces back and forth. The counter-propagating waves interfere with each other and form a standing wave pattern. Modern development of nanofabrication has brought different cavity geometries into place. For example, whispering-gallery-mode resonators (where electromagnetic waves are confined in circular or spherical dielectric structures based on total internal reflection) and photonic crystal nanocavities (where wave confinement is created by introducing defects in periodic photonic crystal lattice structure).



Figure 2.1: A Fabry-Pérot resonator. a, Schematic. b, Normalized amplitude (blue) and phase (red) response.

Here we focus on a traditional Fabry-Pérot resonator, consisting of two highly reflective mirrors that are separated by a distance *L*. Such a resonator supports a series of resonances with angular frequency $\omega_{cav} = n\pi c/L$, with *n* being a positive integer mode number. The frequency separation between two consecutive resonances, or free spectral range (FSR), is defined as:

$$\Delta\omega_{\rm FSR} = \frac{\pi c}{L} \tag{2.14}$$

We will focus on one cavity mode ω_c in most of the derivations throughout this dissertation.

Due to the finite mirror transparencies and the internal absorption/scattering, light in optical cavities will have some decay rate κ^6 . Defining κ_{in} as the decay rate associated with input coupling and κ_0 as the remaining (internal) losses, in general we have

$$\kappa = \kappa_{\rm in} + \kappa_0 \tag{2.15}$$

For Fabry-Pérot resonators, κ_{in} is the transmission loss at the input cavity mirror, and κ_0 summarizes the transmission loss at the second cavity mirror, as well as the scattering and absorption losses in the cavity. Another related quantity called the optical finesse is defined as $\mathcal{F} \equiv \Delta \omega_{FSR}/\kappa$, which corresponds to the average number of roundtrips before a photon leaves the cavity.

A cavity mode coupled to its outside electromagnetic environment can be modeled by a damped harmonic oscillator. Thus for a high-Q cavity mode, we follow the derivation in Subsec. 2.1.1 to describe the intracavity field amplitude with an equation similar to Eq. (2.12), that is:

$$\dot{a}(t) = -(i\omega_{\rm c} + \frac{\kappa}{2})a(t) + \sqrt{\kappa_{\rm in}}a_{\rm in}$$
(2.16)

where *a* is the electromagnetic field amplitude inside the cavity, and $\sqrt{\kappa_{in}}a_{in}$ is the amplitude of the external drive⁷. Eq. (2.16) is essentially a Langevin equation in the classical regime with noise terms ignored. In the quantum regime, *a* and *a*_{in} are replaced by annihilation operators, and terms representing quantum noise are added to make the equation consistent.

If the cavity is driven by a monochromatic laser field $a_{in}(t) = \bar{a}_{in}e^{-i\omega_{L}t}$, we can look for an intracavity response with the form $a(t) = \bar{a}e^{-i\omega_{L}t}$. Making substitutions Eq. (2.16) yields:

$$\bar{a} = \frac{\sqrt{\kappa_{\rm in}}\bar{a}_{\rm in}}{\kappa/2 - i(\omega_{\rm L} - \omega_{\rm c})} = \frac{\sqrt{\kappa_{\rm in}}\bar{a}_{\rm in}}{\kappa/2 - i\Delta}$$
(2.17)

where $\Delta \equiv \omega_{\rm L} - \omega_{\rm c}$ is the detuning of the drive laser frequency with respect to the cavity mode frequency. We sketched the amplitude and phase of the above response \bar{a} in Fig. 2.1(b) as a function

^{6.} We use κ to denote the intensity decay rate, so the amplitude decay rate is $\kappa/2$.

^{7.} Note a_{in} corresponds to $c_{in}/\sqrt{\kappa_{in}}$ in Subsec. 2.1.1. This is because $\hbar\omega_L |a_{in}|^2$ describes the input power, while $\hbar\omega_0 |c_{in}|^2$ scales as the input energy. In the quantum regime, a_{in} is replaced by $\langle \hat{a}_{in} \rangle$, and $|a_{in}|^2$ is replaced by $\langle \hat{a}_{in}^{\dagger} \hat{a}_{in} \rangle$ which represents the rate of photons arriving at the cavity.

of laser detuning Δ . One can verify that the intracavity power (scales with $|\bar{a}|^2$) traces out a Lorentzian line shape with maximum reached at $\Delta = 0$. The intracavity phase evolves by π as the detuning is swept, and near $\Delta = 0$ the change rate is maximized⁸.

According to input-output theory, the field reflected from the cavity is

$$a_{\text{out}} = \sqrt{\kappa_{\text{in}}}a - a_{\text{in}} = -\frac{(\kappa_0 - \kappa_{\text{in}})/2 - i\Delta}{(\kappa_0 + \kappa_{\text{in}})/2 - i\Delta}a_{\text{in}} \equiv \mathcal{R}a_{\text{in}}$$
(2.18)

We can see from this expression that the leakage from the cavity interferes destructively with the prompt reflection, leading to a complex reflection coefficient \mathcal{R} . The ratio between κ_{in} and κ determines the extent of such interference and defines three coupling regimes. As sketched in Fig. 2.2, \mathcal{R} traces out a (counterclockwise) circle as detuning increases from $-\infty$ to $+\infty$. In all three regimes, a Lorentzian-shaped dip will be observed if one measures the power of reflected light while sweeping over laser detuning.



Figure 2.2: Cavity reflection in three coupling regimes.

^{8.} This rapid change is roughly linear, therefore allows cavities to function as sensitive transducers of quantities that are coupled with cavity modes.

2.1.3 Linearized optomechanical interaction

We now consider a canonical optomechanical system that consists of a Fabry-Pérot cavity with one end mirror fixed and the other attached to a spring. The length of the cavity is *L* if there is no light. Once the cavity is pumped with some light source (e.g., a laser resonant with some cavity mode), the intracavity light exerts a radiation pressure force on the end mirrors, and thus modulates the cavity length. If the movable end mirror is displaced by *x*, then the cavity mode frequency is $\tilde{\omega}_c = \frac{1}{1+x/L}\omega_c$. This displacement is usually much smaller than the original cavity length ($x/L \ll 1$, and in this dissertation we have $x/L \sim 10^{-8}$), so we approximately have $\tilde{\omega}_c = \omega_c(1 - x/L)$.



Figure 2.3: Canonical optomechanical system. A generic optomechanical system consists of an optical cavity with a movable boundary, illustrated here as a Fabry-Pérot type resonator in which one mirror is attached to a spring on the wall.

We model the movable end mirror by a damped harmonic oscillator with mass *m* and natural frequency $\omega_{\rm m}$. Defining the zero-point fluctuation $x_{\rm zpf} = \sqrt{\hbar/2m\omega_{\rm m}}$, and recalling that $x = x_{\rm zpf}(c^* + c)$, the intracavity field amplitude is described by:

$$\dot{a} = -(i\omega_{\rm c} + \frac{\kappa}{2})a + ig_0(c^* + c)a + \sqrt{\kappa_{\rm in}}a_{\rm in}$$
(2.19)

where $g_0 = \omega_c(x_{zpf}/L)$ is denoted as the single photon optomechanical coupling rate.

Meanwhile, we can find via electrodynamics the radiation pressure force that drives the end mirror is $F(t) = \hbar \omega_c |a(t)|^2 / L$. Rewrite the equation of motion for the mechanical mode:

$$\dot{c} = -(i\omega_{\rm m} + \frac{\gamma}{2})c + \frac{iF(t)}{\sqrt{2m\hbar\omega_{\rm m}}}$$

$$= -(i\omega_{\rm m} + \frac{\gamma}{2})c + ig_0|a|^2$$
(2.20)

where we have ignored all driving forces except the optical one, to keep the equation simple. In later chapters, we will include the thermal drives for the mechanical modes.

One can tell from the above equations of motion that the optomechanical interaction is fundamentally nonlinear. However, the size of the nonlinearity is set by g_0 , a quantity that is typically small compared with other relevant rates in optomechanical systems. To date, no experiment has observed this nonlinear effect at quantum level (referred to as the single-photon strong-coupling regime). Here we introduce an approximate linearized description that works well as long as $g_0 \ll \min(\kappa, \omega_m)$, which indeed holds for the setup in this dissertation (as well as many other optomechanical systems).

We start the linearization procedure by splitting the intracavity field into an average coherent amplitude *a* and a small fluctuating term *d*:

$$a(t) = \alpha(t) + d(t) \tag{2.21}$$

so that high order terms such as $|d|^2$ in equations of motion may be ignored. Rewriting the system's equations of motion:

$$\dot{\alpha} = -(i\omega_{\rm c} + \frac{\kappa}{2})\alpha + \sqrt{\kappa_{\rm in}}a_{\rm in}$$
(2.22)

$$\dot{d} = -(i\omega_{\rm c} + \frac{\kappa}{2})d + ig_0\alpha(c^* + c)$$
(2.23)

$$\dot{c} = -(i\omega_{\rm m} + \frac{\gamma}{2})c + ig_0(|\alpha|^2 + \alpha d^* + \alpha^* d)$$
(2.24)

The $|\alpha|^2$ term is a constant force that leads to a static displacement, so we ignore it throughout the remainder of this dissertation. The equations can be solved explicitly in the Fourier domain. In order to build more understanding of optomechanical interactions, we (again) suppose the cavity is driven by a monochromatic field at frequency ω_L , and enter a rotating frame that oscillates along with this drive, i.e., $\alpha \to \bar{\alpha}e^{-i\omega_L t}$ and $d \to de^{-i\omega_L t}$, to derive:

$$\bar{\alpha} = \frac{\sqrt{\kappa_{\rm in}}\bar{a}_{\rm in}}{\kappa/2 - i\Delta} \tag{2.25}$$

$$\dot{d} = (i\Delta - \frac{\kappa}{2})d + ig_0\bar{\alpha}(c^* + c)$$
(2.26)

$$\dot{c} = -(i\omega_{\rm m} + \frac{\gamma}{2})c + ig_0(\bar{\alpha}d^* + \bar{\alpha}^*d)$$
 (2.27)

In Hamiltonian mechanics, the time evolution of the system (described by a set of canonical coordinates {*x*, *p*}) is uniquely defined by Hamilton's equations: $\dot{x} = \partial H/\partial q$ and $\dot{q} = -\partial H/\partial x$, where

H = H(x, p, t) is the Hamiltonian that corresponds to the total energy of the system. These equations can be expressed in terms of $\{c, c^*\}$ as $i\dot{c} = \partial H/\partial c^*$ and $i\dot{c^*} = -\partial H/\partial c$. In principle, if we are able to construct a quantity H such that Eq. (2.26) and Eq. (2.27) have forms $i\dot{c} = \partial H/\partial c^*$ and $i\dot{d} = \partial H/\partial d^*$, this quantity is effectively the Hamiltonian of our optomechanical system⁹. As explained in Ch. 1, the energy is not conserved because the optomechanical system (or generally any driven system) is coupled with its environment, resulting in a non-hermitian Hamiltonian:

$$H = -(\Delta + i\frac{\kappa}{2})d^{*}d + (\omega_{\rm m} - i\frac{\gamma}{2})c^{*}c + H_{\rm int}$$
(2.28)

$$H_{\rm int} = -g_0(\bar{\alpha}d^* + \bar{\alpha}^*d)(c^* + c)$$
(2.29)

Next, we qualitatively derive the classical optomechanical effect known as dynamical backaction, where electromagnetic drive tones applied to the cavity can tune the mechanical oscillators' frequencies, dampings, and couplings. Depending on the detuning, three cases ($\Delta < 0$, $\Delta > 0$ and $\Delta = 0$) are distinguished with respect to the optomechanical interaction H_{int} . In the red-detuned case ($\Delta < 0$), H_{int} can be approximated (under RWA) by a beam splitter interaction Hamiltonian $H_{int} \approx g_0(\bar{\alpha}d^*c + \bar{\alpha}^*dc^*)$, and the linearized equations of motion are simplified as:

$$\dot{d} = (i\Delta - \frac{\kappa}{2})d + ig_0\bar{\alpha}c \tag{2.30}$$

$$\dot{c} = -(i\omega_{\rm m} + \frac{\gamma}{2})c + ig_0\bar{\alpha}^*d \tag{2.31}$$

These equations can easily be solved if we assume the cavity field follows the mechanical motion instantaneously, i.e., $d = (\kappa/2 - i\Delta)^{-1} ig_0 \bar{\alpha} c^{10}$. Eliminate the cavity field *d* in 2.31 and we have

$$\dot{c} = -i\left(\omega_{\rm m} - \frac{g_0^2 |\bar{\alpha}|^2 |\Delta|}{(\kappa/2)^2 + \Delta^2}\right)c - \frac{1}{2}\left(\gamma + \frac{g_0^2 |\bar{\alpha}|^2 \kappa}{(\kappa/2)^2 + \Delta^2}\right)c$$

$$= -i(\omega_{\rm m} + \delta\omega_{\rm r})c - \frac{1}{2}(\gamma + \delta\gamma_{\rm r})c$$
(2.32)

where $\delta \omega_r$ and $\delta \gamma_r$ are the optical spring and damping terms, respectively. One may observe that in the red-detuned regime, the mechanical oscillator is softened ($\delta \omega_r < 0$) and is more dissipative

^{9.} This is a pedagogical way to derive the effective Hamiltonian in the classical regime. Rigorous treatment that would apply to the quantum regime can be found in Ref. [113]

^{10.} Note that this is a heuristic derivation in the Doppler regime where the cavity decay is large.

 $(\delta \gamma_r > 0).$

In the blue-detuned case ($\Delta > 0$), the interaction term is approximately a two-mode squeezing Hamiltonian: $H_{\text{int}} \approx g_0(\bar{\alpha}d^*c^* + \bar{\alpha}^*dc)$. We follow the same procedure as above to eliminate the cavity field and derive:

$$\dot{c} = -i\left(\omega_{\rm m} + \frac{g_0^2 |\bar{\alpha}|^2 |\Delta|}{(\kappa/2)^2 + \Delta^2}\right)c - \frac{1}{2}\left(\gamma - \frac{g_0^2 |\bar{\alpha}|^2 \kappa}{(\kappa/2)^2 + \Delta^2}\right)c$$

$$= -i(\omega_{\rm m} + \delta\omega_{\rm b})c - \frac{1}{2}(\gamma + \delta\gamma_{\rm b})c$$
(2.33)

We find $\delta\omega_b = -\delta\omega_r$ and $\delta\gamma_b = -\delta\gamma_r$ by comparing 2.32 and 2.33. The mechanical spring constant is stiffened ($\delta\omega_b > 0$) and anti-damping ($\delta\gamma_b < 0$) is induced in the blue-detuned regime. If the total mechanical damping $\gamma + \delta\gamma_b$ becomes negative, the oscillation amplitude c(t) increases exponentially. The linearization will break down when c(t) is large enough, so the oscillation amplitude will be limited by nonlinear effects, such that stable self-sustained oscillations can develop [113].

Finally, when the drive is on resonance with the cavity ($\Delta = 0$), the mechanical position $x = x_{ZPF}(c^* + c)$ causes a phase shift of the light field, which is encountered in optomechanical displacement measurements. In quantum mechanics, the fact that \hat{x} (i.e., $\hat{a} + \hat{a}^{\dagger}$) commutes with the interaction Hamiltonian enables the quantum non-demolition (QND) measurement on the optical amplitude quadrature [121].

It is worth mentioning that the linearized equations of motion have an exact solution in the frequency domain [113], which holds for both resolved-sideband regime and Doppler regime (where we have a "bad cavity" such that $\kappa \gg \omega_m$). In this solution, the optical spring $\delta\Omega_m$ and damping $\delta\Gamma_m$ are expressed as modifications of the mechanical resonator's linear response to an external force, and in general being frequency dependent:

$$\delta\Omega_{\rm m}(\omega) = g_0^2 |\alpha|^2 \frac{\omega_{\rm m}}{\omega} \left[\frac{\Delta + \omega}{(\Delta + \omega)^2 + \kappa^2/4} + \frac{\Delta - \omega}{(\Delta - \omega)^2 + \kappa^2/4} \right]$$
(2.34)

$$\delta\Gamma_{\rm m}(\omega) = g_0^2 |\alpha|^2 \frac{\omega_{\rm m}}{\omega} \left[\frac{\kappa}{(\Delta + \omega)^2 + \kappa^2/4} - \frac{\kappa}{(\Delta - \omega)^2 + \kappa^2/4} \right]$$
(2.35)

Note $\delta\Omega_{\rm m}$ and $\delta\Gamma_{\rm m}$ can be evaluated at the original, unperturbed mechanical frequency $\omega = \omega_{\rm m}$, as long as the laser drive is sufficiently weak $(g_0|\alpha| \ll \kappa)$. We sketch $\delta\omega_{\rm m} = \delta\Omega_{\rm m}(\omega_{\rm m})$ and $\delta\gamma_{\rm m} = \delta\Gamma_{\rm m}(\omega_{\rm m})$ in Fig. 2.4.



Figure 2.4: Dynamical backaction in three regimes. (a) $\kappa = 0.1\omega_m$, (b) $\kappa = \omega_m$, (c) $\kappa = 10\omega_m$. We have set $g_0 = 1$ Hz, $\sqrt{\kappa_{in}}|a_{in}| = 1$ MHz, $\omega_m = 1$ kHz.

By the end of this part, we emphasize that the aforementioned linearization process can be applied to a multi-mode optomechanical system. For example, the Hamiltonian of an *m*-optical-*n*-mechanical-mode system driven by a monochromatic laser field can be written as:

$$H = -\sum_{j=1}^{m} (\Delta_j + i\frac{\kappa_j}{2}) d_j^* d_j + \sum_{k=1}^{n} (\omega_{m,k} - i\frac{\gamma_{m,k}}{2}) c_k^* c_k + H_{\text{int}}$$
(2.36)

$$H_{\text{int}} = -\sum_{j=1}^{m} \sum_{k=1}^{n} g_{jk} (\bar{\alpha}_j d_j^* + \bar{\alpha}_j^* d_j) (c_k^* + c_k)$$
(2.37)

where g_{jk} is the coupling rate between the *j*th optical mode and the *k*th mechanical mode.

2.1.4 Membrane-in-the-middle system

Experimental implementations of cavity optomechanical systems vary widely in scale and geometry. In this dissertation we focus on the membrane-in-the-middle (MIM) geometry, where a thin dielectric membrane is dispersively coupled to a Fabry-Pérot cavity (Fig. 2.5). This geometry separates the mechanical oscillator from the cavity mirrors, and is more favorable than the canonical optome-chanical system (Fig. 2.3), because the cavity mirrors are no longer required to be highly reflective, mechanically compliant, and of high mechanical quality factor as would be in the canonical system.



Figure 2.5: Membrane-in-the-middle setup. **a**, Schematic of a partially reflective membrane inside a Fabry-Pérot cavity. **b**, Theoretical plot of perturbed cavity resonance frequency as a function of membrane position normalized to wavelength. Blue, orange, green, red lines correspond to membrane's reflectivity at 0.0, 0.65, 0.8, 0.95, respectively.

The presence of the membrane perturbs the intracavity electromagnetic field. Since the thickness of the membrane is usually much less than the intracavity electromagnetic wavelength, such perturbation can be adjusted by moving the membrane within the cavity. For example, the perturbation is minimal when the membrane is located at a node of the intracavity standing wave. On the other hand, maximal perturbation is reached when the membrane moves to a cavity antinode, where the refraction inside the membrane now effectively increases the cavity length, and thus resulting in a lower-shifted cavity resonance frequency.

To quantitatively describe how the membrane affects the cavity resonances, we present a mathematical solution of the perturbed fundamental Gaussian mode frequency. For simplicity, we assume the membrane is located near the optical beam waist and is oriented perpendicularly to the optical mode's wave vector. The membrane is modeled as a dielectric slab with thickness L_d and refraction index *n*. Its reflectivity and transmissivity for a light with wave number *k* at normal incidence are [122]:

$$r_{\rm d} = \frac{(n^2 - 1)\sin knL_{\rm d}}{2in\cos knL_{\rm d} + (n^2 + 1)\sin knL_{\rm d}}$$

$$t_{\rm d} = \frac{2n}{2in\cos knL_{\rm d} + (n^2 + 1)\sin knL_{\rm d}}$$
(2.38)

Denote the reflectivity (transmissivity) of cavity end mirrors with $r_{1,2}$ ($t_{1,2}$), the field amplitudes inside the cavity (Fig. 2.5a) are constrained by boundary conditions:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} 0 & r_1 e^{ikL_1} & 0 & 0 \\ r_d e^{ikL_1} & 0 & 0 & t_d e^{ikL_2} \\ t_d e^{ikL_1} & 0 & 0 & r_d e^{ikL_2} \\ 0 & 0 & r_2 e^{ikL_2} & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$
(2.39)

where L_1 (L_2) is the length of the left (right) half of the cavity, such that $L_1 - L_2 = 2x$, $L_1 + L_2 = L$. Basic linear algebra shows that Eq. (2.39) has nontrivial solutions only when the coefficient matrix on the RHS is similar to an identity matrix. Our primary interest is in the case of cavity mirrors being highly reflective, i.e., $r_1 \approx r_2 \approx 1$. Setting all eigenvalues of the coefficient matrix equal to 1 gives the cavity resonance frequency:

$$\omega(x) = \Delta\omega_{\text{FSR}}(\cos^{-1}(|r_{d}|\cos 2kx) + \phi) \cdot \frac{2}{\pi}$$
(2.40)

where $\phi = \arg r_d$ is the complex phase of r_d , and again $x \ll L$ holds. We plot $\omega(x)$ in Fig. 2.5 with $|r_d|$ set to several different values. Note that neither higher-order modes of the cavity nor tilted alignment of the membrane was taken into account in our current treatment. General treatment of these effects based on perturbation theory is given in Ref. [123].

2.2 Optomechanical nonreciprocal photon transmission

We explained in Ch. 1 that nonreciprocity can be induced from parametric modulations in time. For a parametrically coupled multi-mode system, nonreciprocity may arise due to the dissipation in ancillary modes along with the interference between multiple coupling pathways [124]. In this context, dissipation allows a flow of energy leaving the system, thus breaking the time reversal symmetry, regardless of the interchange between inputs and outputs, and can make any coherent interaction directional (can be viewed as reservoir engineering, see in Ref. [125]). Multi-path interference, if dependent on the direction of signal propagation, is another key resource to break reciprocity. For example, transmission in one direction is allowed due to constructive interference between different paths (see in Ch. 1 for details), while destructive interference suppresses the transmission in the opposite direction.

Following this approach, optomechanical systems are particularly promising for parametric nonreciprocity. On the one hand, parametric modulations of mechanical and/or optical modes are naturally realized in an optomechanical setting, thanks to the dynamical backaction. On the other hand, the system is intrinsically dissipative, while the interference needed for nonreciprocity may be achieved by combining independent optomechanical interactions. Indeed, theories have been presented to describe possible optomechanical implementations of on-chip isolators [126–129], frequency converters [130], and directional amplifiers [131]. Meanwhile, a bunch of experiments have demonstrated electromagnetic nonreciprocity in both optical [128,129,132–134] and microwave [52, 135–137] frequencies.

In this section, I will present the theoretical framework for general conditions to induce nonreciprocal transmission, and show that these conditions can be satisfied with multi-mode optomechanical arrangements. Then I will summarize the results from several experimental realizations.

2.2.1 Theoretical proposals

Consider a general optical two-port two-mode system (sketched in Fig. 2.6), which can be described with coupled-mode formalism [138]:

$$i\dot{\mathbf{a}} = D\mathbf{a} + K\mathbf{s}_{\rm in} \tag{2.41}$$

$$\mathbf{s}_{\text{out}} = P\mathbf{s}_{\text{in}} + K^{\mathrm{T}}\mathbf{a} \tag{2.42}$$

where $\mathbf{a} = (a_1, a_2)^{\mathrm{T}}$ is the vector amplitude of the two modes, and \mathbf{s}_{in} ($\mathbf{s}_{\mathrm{out}}$) represents the input (output) signals at the two ports. The dynamical matrix *D* describes the linear evolution of the two-mode subsystem in absence of excitation, while the *P* is the direct path scattering matrix between

the two ports¹¹. Matrices K and K^T represent the coupling from ports to modes and from modes to ports, separately. Note that we have implicitly used the convention that each optical mode couples to the input/ouput port in a reciprocal fashion.



Figure 2.6: Schematic representation of a two-port two-mode system. Couplings within the system are denoted with double-headed arrows. Dashed lines indicate the coupling may not exist. The ports located at two ends can be used for input and/or output. Each mode (a_1 or a_2) couples with both ports.

In the frequency domain, Eq. (2.41) and Eq. (2.42) are rewritten as:

$$\omega \mathbf{a}[\omega] = D(\omega)\mathbf{a}[\omega] + K\mathbf{s}_{\text{in}}[\omega]$$
(2.43)

$$\mathbf{s}_{\text{out}}[\omega] = P\mathbf{s}_{\text{in}}[\omega] + K^{\mathrm{T}}\mathbf{a}[\omega]$$
(2.44)

We can define the scattering matrix $S(\omega)$ of the system such that $\mathbf{s}_{out}[\omega] = S(\omega)\mathbf{s}_{in}[\omega]$. The offdiagonal matrix element S_{12} (S_{21}) represents forward (backward) transmission coefficient. The scattering matrix solved from Eq. (2.43) and Eq. (2.44) is:

$$S(\omega) = P + K^{\mathrm{T}}(\omega I - D(\omega))^{-1}K$$
(2.45)

The system is nonreciprocal if the forward and backward transmissions are different, which can be quantified by

$$S_{12} - S_{21} = \frac{\det(K)(D_{12} - D_{21})}{\det(\omega I - D)}$$
(2.46)

According to 2.46, the necessary and sufficient conditions to break reciprocity are (i) det(P) \neq 0, (ii) $D_{12} \neq D_{21}$. Condition (i) can easily be ensured with a suitable asymmetry in the coupling between the two modes and the two ports, namely $K_{12}K_{21} \neq K_{11}K_{22}$. Condition (ii) is more demanding, as the Hamiltonian of a linear, time-invariant, time-reversible system is always symmetric [5]. We now

^{11.} Here P is a unitary matrix, so $|\det(P)| = 1$, while the phase of $\det(P)$ is in general not measurable. P is diagonal if there is no direct coupling between the two ports.

show how such symmetry can be broken with optomechanical interactions.



Figure 2.7: Schematic of nonreciprocal optomechanical systems. **a**, A system consists of two optical modes coupled to one mechanical mode, adapted from Ref. [139]. Interactions with the common mechanical mode create a mechanically mediated coupling between the two optical modes, which is, in general, nonreciprocal. **b**, A system consists of two optical modes and two mechanical modes, adapted from Ref. [130]. Nonreciprocity in amplitude arises due to multi-path interference.

Let us focus on photon transmissions in a cavity-optomechanical system that involves two optical modes coupled to a common mechanical mode (Fig. 2.7(a)). The natural frequency and linewidth of the mechanical mode are ω_m and γ_m , respectively. A control field at frequency ω_L is applied to both optical modes. In the rotating frame at frequency ω_L , the linearized evolution of this system is described by:

$$\alpha_j = \frac{\sqrt{\kappa_{\text{in},j}}}{\kappa_j/2 - i\Delta_j} a_{\text{in},j}$$
(2.47)

$$\dot{d}_j = (i\Delta_j - \frac{\kappa_j}{2})d_j + ig_j\alpha_j(c+c^*)$$
(2.48)

$$\dot{c} = -(i\omega_{\rm m} + \frac{\gamma_{\rm m}}{2})c + \sum_{j=1}^{2} ig_j(\alpha_j^* d_j + \alpha_j d_j^*)$$
(2.49)

where $\Delta_j = \omega_L - \omega_{c,j}$ (j = 1, 2) defines the detuning of the control field frequency with respect to each optical mode, and κ_i describes the optical mode's decay rate. Note in Eq. (2.47) we have implicitly assumed that each optical mode is driven only by the control field in its own cavity (that no direct coupling between the two optical modes are introduced), which corresponds to the assumption that *K* is diagonal in the aforementioned general two-port systems. We also assume the system is in the resolved-sideband regime and the control field is red-detuned, such that the counter-rotating terms (those contain c^* or d^*) in Eq. 2.48 and Eq. 2.49 can be ignored. After eliminating the mechanical motion in the frequency domain, we get:

$$\begin{pmatrix} \left[\frac{\kappa_1}{2} - i(\Delta_1 + \omega)\right] d_1\\ \left[\frac{\kappa_2}{2} - i(\Delta_2 + \omega)\right] d_2 \end{pmatrix} = -\chi_{\rm m}(\omega) \begin{pmatrix} g_1^2 |\alpha_1|^2 & g_1 g_2 \alpha_1 \alpha_2^*\\ g_1 g_2 \alpha_1^* \alpha_2 & g_2^2 |\alpha_2|^2 \end{pmatrix} \begin{pmatrix} d_1\\ d_2 \end{pmatrix}$$
(2.50)

where the mechanical susceptibility is defined as $\chi_m(\omega) = [\gamma_m/2 - i(\omega - \omega_m)]^{-1}$. Comparing Eq. 2.50 with Eq. 2.43, it follows that

$$D(\omega) = \begin{pmatrix} -\Delta_1 - i\frac{\kappa_1}{2} & 0\\ 0 & -\Delta_2 - i\frac{\kappa_2}{2} \end{pmatrix} - i\chi_{\rm m}(\omega) \begin{pmatrix} g_1^2 |\alpha_1|^2 & g_1g_2\alpha_1\alpha_2^*\\ g_1g_2\alpha_1^*\alpha_2 & g_2^2 |\alpha_2|^2 \end{pmatrix}$$
(2.51)

We observe from Eq. (2.51) that in general $D_{12} \neq D_{21}$ since α_1 and α_2 are complex numbers. Therefore the system can be nonreciprocal, by imprinting opposite phase for oppositely traveling photons. However, the fact that $|D_{12}| = |D_{21}|$ indicates the transmission amplitude remains the same for both input ports/directions. To achieve nonreciprocal amplitude transmission (for application in isolators), we may introduce the direct coupling μ between the two optical modes, such that the new dynamical matrix is $D(\mu) = D + \mu \sigma_x$ (where σ_x is the Pauli matrix). With appropriate choice of μ , we would have $|D(\mu)_{12}| \neq |D(\mu)_{21}|$ [139].

Alternatively, we may introduce another photon transmission path by including a second mechanical degree of freedom (Fig. 2.7b) and a second red-detuned control field. Interference occurs as there are two transmission paths between the cavity modes. Denote the mechanical frequencies and dampings by $\omega_{m,k}$ and γ_k for k = 1, 2. The control laser frequencies are $\omega_{L,1}$ and $\omega_{L,2}$, and the detunings are $\Delta_{ij} = \omega_{L,i} - \omega_{c,j}$. Linearize the cavity fields with $a_j = e^{-i\omega_{c,j}t} (\sum_{i=1}^2 \alpha_{ij}e^{-i\Delta_{ij}t} + d_j)$, where $\alpha_{ij} = \frac{\sqrt{\kappa_{m,i}}}{\kappa_i/2 - i\Delta_{ij}}$, and write down the equations of motion in the resolved-sideband regime:

$$\dot{d}_j = -\frac{\kappa_j}{2} d_j + \sum_{k=1}^2 i g_{jk} \alpha_{jk} (c_k + c_k^*)$$
(2.52)

$$\dot{c}_{k} = -\left(i\omega_{m,k} + \frac{\gamma_{k}}{2}\right)c_{k} + \sum_{j=1}^{2}ig_{jk}\alpha_{jk}^{*}d_{j}$$
(2.53)

from which the mechanical modes can be eliminated to derive the dynamical matrix. By entering a

proper rotating frame, we have:

$$D(\omega) = \begin{pmatrix} \delta_r - i\frac{\kappa_1}{2} & 0\\ 0 & -\delta_r - i\frac{\kappa_2}{2} \end{pmatrix} + \sum_{k=1}^2 \chi_{m,k} (\omega + (2k-3)\delta_r) G_k$$
(2.54)

where $\delta_r = (\omega_{L,1} - \omega_{L,2})/2$ describes the frequency difference between the two control fields, $\chi_{m,k}(\omega) = (\gamma_k/2 - i(\omega - \omega_{m,k}))^{-1}$ denotes the mechanical susceptibility, and the interaction matrix G_k has elements $(G_k)_{ij} \equiv G_{kij} = g_{ik}g_{jk}\alpha_{ik}\alpha_{jk}^*$ for k = 1, 2. The ratio between backward and forward transmission can be shown to have form:

$$\frac{S_{12}(\omega)}{S_{21}(\omega)} = \frac{G_{112}\chi_{m,1}(\omega - \delta_r) + G_{212}\chi_{m,2}(\omega + \delta_r)}{G_{121}\chi_{m,1}(\omega - \delta_r) + G_{221}\chi_{m,2}(\omega + \delta_r)}$$
(2.55)

As we will see below, this ratio is demonstrated to have a magnitude that deviates from 1 in various experimental settings.

We are left with a remark on the off-diagonal terms in Eq. (2.51) and Eq. (2.54), which represent the effective coupling between the two optical cavity modes mediated by the mechanical mode(s). As we know, the mechanical mode(s) can be driven by the beating between the light on resonance with one cavity and its associated control field(s), while such mechanical motion adds a blue sideband to any control field that is applied to the other cavity mode. As long as this sideband is approximately on resonance with the second cavity mode (which mathematically corresponds to $|\omega_{c,1} - \omega_{c,2}| < \min(\kappa_1, \kappa_2))$, we have a decent coupling between the two cavity modes. In practice, however, we may have distinct cavity frequencies. Therefore additional control field(s) should be introduced in order to effectively couple the cavity modes. Generally speaking, an optomechanical system with *m* optical cavity modes and *n* mechanical modes requires $m \times n$ control fields to generate nonreciprocity in amplitude transmission¹².

2.2.2 Experimental implementations

We now review the main results from a few experimental work that demonstrate nonreciprocal photon transmission based on optomechanical interactions. Both two-optical-one-mechanical-mode and two-optical-two-mechanical-mode systems are discussed. This helps us build more understanding of

^{12.} We assume the cavity modes do not directly interact with each other.

the theory presented in the previous section.



Figure 2.8: Nonreciprocity in microsphere resonators. Summary of results from Ref. [133].

As reported in Ref. [133], nonreciprocity was induced in a traveling wave optomechanical system, which consisted of a silica microsphere resonator that was evanescently coupled with a tapered fibre. Three modes of the resonator, namely a pair of degenerate CW and CCW optical whispering gallery modes and the mechanical breathing vibration mode (where the equator of the sphere expands and compresses uniformly) were dispersively coupled by optomechanical interaction. A driving field enhanced this optomechanical interaction between the mechanical mode and the copropagating optical mode inside the microsphere resonator. Specifically, a transparency transmission window arose for the copropagating signal when the driving field was red-detuned from the optical mode

frequency¹³, leading to optical isolation (Fig. 2.8c, d). On the other hand, a blue-detuned driving field enhanced the copropagating signals, resulting in a directional amplifier (Fig. 2.8e, f). The authors also studied the nonreciprocal transmission in the case of two oppositely propagating driving fields (Fig. 2.8g, h).

Similar isolation and directional amplification of optical signals were realized with silica microtoroid optomechanical resonators [128]. The nonreciprocal transmission was shown to be preserved even for non-degenerate modes. In a subsequent work by the same group [132], the authors demonstrated a four-port circulation with almost identical setup (except including an additional tapered optical fiber), as illustrated in Fig. 2.9. The isolator built with microsphere resonators was also reported [134].



Figure 2.9: Nonreciprocity in microtoroid resonators. Summary of results from Ref. [132].

^{13.} This phenomena is usually referred to as the optomechanically induced transparency (OMIT). Its counterpart for the blue-detuned control field is called the optomechanically induced amplification (OMIA).

Nonreciprocity due to multi-path interference has mostly been reported in the microwave domain, and demonstrated with superconducting circuits [140]. For example, in Ref. [136], two cavity modes (produced by a vacuum-gap capacitor resonant with spiral inductors) were coupled with two mechanical modes (generated from the vibrations of the capacitor's top plate) via four independent drives (Fig. 2.10(a)). With this configuration, nonreciprocal transmission can be adjusted to occur on resonance with the cavities, by driving at appropriate amplitudes and frequencies. The relative phases of the drives also played an important role for maximizing the nonreciprocity. It turned out that the so called loop phase, which defined as a dynamically tunable parameter related to the sum of the relative phases of the four drives, can be optimized to realize a reverse isolation of more than 20 dB (Fig. 2.10(b)).



Figure 2.10: Nonreciprocity in superconducting circuits. Summary of results from Ref. [136].

Authors in another study [52] introduced a third cavity mode to demonstrate a mechanically mediated circulator. To realize three-port circulation, one requires at least six distinct photon-phonon conversion paths. This was implemented by a set of six phase-locked microwave pumps. An isolation up to 24–38 dB was reported.

Last but not least, nonreciprocity was also highlighted in a photonic-crystal-based optomechanical circuit [129]. In this experiment, two cavities were connected both optically and mechanically. As a result, there is a direct coupling between the two cavity modes, along with the "indirect" coupling mediated by the two mechanical modes. When the cavities were driven by the phase-correlated control beams, a synthetic magnetic magnetic flux formed, which, in combination with the system's dissipation, led to nonreciprocal transmission of photons with 35 dB of isolation.

2.3 Optomechanical nonreciprocal phonon transmission

We showed in the previous section how two otherwise non-interacting optical cavity modes can be effectively coupled together in the presence of optomechanical interactions. The linearized optomechanical Hamiltonian, proportional to $\alpha dc^* + \alpha^* d^*c$ ($\alpha d^*c^* + \alpha^* dc$) when the control field is red- (blue-) detuned, is symmetric under the exchange of the optical and mechanical degrees of freedom. It is thus natural to think about the dual form of the proposals in Sec. 2.2. That is, instead of using mechanical modes as the auxiliary to produce nonreciprocal photon transmission, we may think of producing nonreciprocity in the transmission of the mechanical excitation (i.e., phonon) with the optical mode(s) as the auxiliary.



Figure 2.11: Schematic of (non)reciprocal optomechanical phonon transmission. Based on the symmetry between mechanical and optical degrees of freedom in Fig. 2.7(b), we may consider the nonreciprocity in mechanical modes. We further save one optical mode by introducing multi-path optomechanical coupling between the mechanical modes and the same cavity resonance.

As an analogy of the two-optical-one-mechanical-mode system discussed in Sec. 2.2, consider a system consisting of two mechanical modes (with natural frequencies $\omega_{1,2}$ and linewidths $\gamma_{1,2}$), each coupled linearly to a common optical cavity mode (with resonance frequency ω_c and decay rate κ). Suppose the cavity is driven by a control laser beam with power *P* and frequency ω_L , the linearized equations of motion are given by:

$$\dot{d} = (i\Delta - \frac{\kappa}{2})d + i\sum_{k=1}^{2} g_k \alpha (c_k + c_k^*)$$
(2.56)

$$\dot{c}_k = -(i\omega_{\mathrm{m},k} + \frac{\gamma_k}{2})c_k + ig_k(\alpha^*d + \alpha d^*)$$
(2.57)

where $\Delta = \omega_{\rm L} - \omega_{\rm c}$ is the control laser detuning, and average intracavity field is:

$$\alpha = \frac{\sqrt{\kappa_{\rm in}}}{\frac{\kappa}{2} - i\Delta} a_{\rm in}; \quad a_{\rm in} \equiv \sqrt{\frac{P}{\hbar\omega_{\rm L}}}$$
(2.58)

The motion in the reduced system of two mechanical modes is described by a vector $\mathbf{c} = (c_1, c_2)^T$, which in the frequency domain satisfies

$$-i\omega\mathbf{c}[\omega] = -\begin{pmatrix} \frac{\kappa_1}{2} + i\omega_{m,1} & 0\\ 0 & \frac{\kappa_2}{2} + i\omega_{m,2} \end{pmatrix} \mathbf{c}[\omega] + \begin{pmatrix} g_1^2 & g_1g_2\\ g_1g_2 & g_2^2 \end{pmatrix} \sigma\mathbf{c}[\omega]$$
(2.59)

where the optical susceptibility is defined as $\chi_c(\omega) = [\kappa/2 - i(\omega + \Delta)]^{-1}$, and the complex mechanical susceptibility induced by the intracavity field is $\sigma = |\alpha|^2 [\chi_c(\omega) - \chi_c^*(-\omega)]$.

Since the coefficient matrix on the RHS of Eq. (2.59) is symmetric, static phonon transmission between the two mechanical modes is reciprocal. However, the phononic energy transfer may be nonreciprocal if the parameters of the optomechanical system are varied in time. Such nonreciprocity was first demonstrated in our MIM system with the presence of an EP in the spectrum of the device [141].

Here we briefly summarize the idea and results of our work in Ref. [141], and leave the experimental details (of this work as well as our subsequent studies in Ref. [142, 143]) for later chapters. In this study, a pair of nearly-degenerate mechanical modes were coupled via a single optical cavity driven by monochromatic red-detuned control laser beam. We showed that with appropriately chosen control laser power (*P*) and detuning (Δ), an EP appears in the mechanical-mode subsystem (Fig. 2.12a–d). In addition, we were able to execute closed loops that encircle the EP in (*P*, Δ) parameter space in a time not much greater than the mechanical modes' decay times. To study the phononic energy transfer, we excited one of the mechanical modes, then executed the loop, and recorded the motion in both mechanical modes right after the loop was done. The efficiency of the loop's energy



transfer was calculated from the mechanical motion measurement.

Figure 2.12: Summary of results in Ref. [141]. **a**–**d**, The resonance frequencies and damping rates of the two mechanical modes of the membrane as a function of laser power P and detuning Δ , demonstrating the existence of an EP. **e**, Magnitudes of propagator matrix elements. Results of clockwise (counterclockwise) control loops are shown in circles (squares). For each control loop direction, the inequality of the red/blue markers in the long-time limit indicate the non-reciprocity of the propagator matrix.

The energy transfer was negligible when the loop was swept out very quickly. In contrast,

nonreciprocal energy transfer was observed in the adiabatic limit (loop time $\tau \gg 1$ ms in this experiment), for both clockwise and counter-clockwise loops. Mathematically, an adiabatic control loop can be viewed as a propagator matrix that transforms the initial state $\mathbf{c}(0) = (c_1(0), c_2(0))^T$ to the final state $\mathbf{c}(\tau) = (c_1(\tau), c_2(\tau))^T$, with the form:

$$U_{\cup,\cup}(\tau) = \begin{pmatrix} a_{\cup,\cup}(\tau) & b_{\cup,\cup}(\tau) \\ c_{\cup,\cup}(\tau) & d_{\cup,\cup}(\tau) \end{pmatrix}$$
(2.60)

where $\bigcup (\bigcup)$ denotes a (counter-) clockwise loop. Nonreciprocity in the energy transfer occurs when $b_{\bigcup,\bigcup}(\tau) \neq c_{\bigcup,\bigcup}(\tau)^{14}$, which is shown in Fig. 2.12e.

In a subsequent study [142], we observed similar nonreciprocal energy transfer within a pair of highly non-degenerate mechanical modes. As will be discussed in Ch. 4, effective coupling between these highly non-degenerate mechanical modes is achieved by driving the cavity with two control laser tones, in which case the eigenvalue spectrum has similar features to the nearly-degenerate case. In the high-Q limit, such arrangement leads to a dynamical matrix D of the mechanical two-mode subsystem that satisfies $D_{12} \approx D_{21}^*$, indicating nonreciprocity in phase. The energy transfer is nonreciprocal for the adiabatic encircling of an EP.

It is worth mentioning that the nonreciprocity studied in Ref. [141, 142] is transient, as we necessarily had to modulate system parameters dynamically in time. In Ch. 5, we will present measurements of static optomechanical nonreciprocity, in which no time-dependent modulation is required. The basic idea is to introduce another pair of cavity drives, such that the off-diagonal elements of the dynamical matrix is highly asymmetric $(|D_{12}| \neq |D_{21}|)$.

^{14.} If we map this transfer process to the scattering through a two-mode waveguide, such that the sense of the loop corresponds to the direction of wave propagation, then the 4×4 scattering matrix is symmetric so the system is reciprocal. A detailed explanation is in Appendix A.



Experimental setup

The optomechanical system for our experiments consists of a silicon nitride membrane positioned in the middle of a Fabry-Pérot cavity. With the theoretical model already discussed in Ch. 2, in this chapter I focus on the experimental setup. I will review the MIM device (including the membrane and the cavity), the cryogenic environment, the optical and electronic circuits, as well as the laser locking techniques for our experiment.

3.1 Cryogenic membrane-in-the-middle system

In this section, I describe the membrane and the cavity setup, and how the MIM device is settled down in the cryogenic platform.

3.1.1 Membrane

In general, the vibrational modes of a 2D (or very thin and highly stressed) square membrane can be indicated by the number of anti-nodes in its two dimensions (Fig. 3.1). For instance, a $\{m, n\}$ mode describes a standing wave at frequency

$$\nu_{m,n} = \nu_{1,1} \sqrt{\frac{m^2 + n^2}{2}} \tag{3.1}$$

where $v_{1,1}$ is the membrane's fundamental mode frequency. We can see that in this case $\{m, n\}$ and $\{n, m\}$ are degenerate modes. For a rectangular membrane with unequal side lengths (i.e., $L_x \neq L_y$), the degeneracy between $\{m, n\}$ and $\{n, m\}$ modes is broken, and the frequency of the $\{m, n\}$ mode is now

$$v_{m,n} = v_{1,1} \sqrt{\frac{(m/L_x)^2 + (n/L_y)^2}{(1/L_x)^2 + (1/L_y)^2}}$$
(3.2)



Figure 3.1: Vibrational patterns of a square membrane.

Commercial silicon nitride membranes are known for their high mechanical quality factors and low optical absorption in the near-infrared regime [144], which make them attractive candidates for the mechanical resonators of the MIM geometry. For our experiment, we use a high stress, 1 mm × 1 mm × 50 nm stoichiometric Si_3N_4 membrane from Norcada (NX5100AS).

The frequency of the fundamental mode on our membrane is approximately 352 kHz. The ideal frequencies of high order $\{m, n\}$ modes can be inferred via Eq. (3.1). In practice, the membrane is not a perfect square due to imprecision in the fabrication process. However, as separation between $\{m, n\}$

and $\{n, m\}$ modes is typically several hundred Hertz (i.e., much smaller than the splitting between mode pairs which are not $\{m, n\}$ and $\{n, m\}$), we refer to such mode pairs as nearly-degenerate. A sample spectrum of our membrane is shown in Fig. 3.2.



Figure 3.2: Membrane mode spectrum. Red markers indicate the theoretical values calculated from Eq. (3.1) given $v_{1,1} = 352.4$ kHz. The unmarked smaller peaks are other vibrational modes of the system (could be of the cavity mirrors or membrane support structures).

At room temperature, the Q factors of these modes are around 10^6 , and can be increased to 10^7 as the temperature goes to 4K. We measure the Qs via mechanical ringdowns (see in App. B, where the setup is discussed and several examples of such measurement are shown).



Figure 3.3: Membrane setup. a, Schematic of the membrane support. b, Photograph of the membrane support.

To insert the membrane into the optical cavity, we mount it on a metal-based multilayer support. As shown in Fig. 3.3, the silicon chip of the membrane is held on a circular plate of oxygen-free high-conductivity (OFHC) copper, beneath which lies another sheet of copper that is thermally anchored to the experimental stage¹. The Si chip and copper plates are screwed onto a rectangular titanium block, which is mounted on top of a ring piezo actuator (only wires visible in Fig. 3.3). Finally the ring piezo sits on a titanium "bridge" that supports the whole assembly.

^{1.} Heat sinking wires are pressed onto the copper sheet to create a good thermal link to the cryostat.

3.1.2 Cavity

Our optical cavity consists of two high-reflectivity mirrors that are mounted on two ends of a 3.7 cm titanium spacer. These mirrors (purchased from ATFilms) are formed by a stack of alternating dielectric coatings deposited on a glass mirror substrate, and are clamped between two plates that are held together by screws with spring washers. (see in Fig. 3.4) The spring washers ensure an even clamping force on the mirrors, even when the thermal contraction between the mirror substrate and the screws is non-uniform.



Figure 3.4: Cavity setup. a, Schematic of the cavity. b, Photograph of the cavity with membrane in place.

In our current setup, the cavity output mirror (r = 0.99997) is significantly more reflective than the input mirror (r = 0.9998), such that most of the light that leaks out of the cavity traces back to the input path, making the cavity almost single-sided. The 3.7 cm cavity length corresponds to a free-spectral range ~ 4 GHz, while the measured cavity decay rate and coupling rate are $\kappa = 180$ kHz and $\kappa_{in} = 70$ kHz, separately. The decay rate can be measured via cavity ringdown (see App. B), and the coupling rate can then be determined by the cavity reflection dip. Another important feature we need to measure is the optomechanical coupling rate g_0 , which is done by fitting measurements of the optical spring and damping effects (see App. B).

3.1.3 Cryostat



Figure 3.5: Schematic of the major components of a He-3 cryostat.

The cryogenic environment is provided by a Janis ³He refrigerator. As illustrated in Fig. 3.5, our cryostat consists of a ³He insert situated in the inner vacuum chamber (IVC), which is submerged in a vacuum jacketed liquid ⁴He bath. The ⁴He bath (with constant temperature ~ 4 K) is vented to atmosphere (or a helium recycling system) and is replenished approximately weekly by transferring from an external source. Inside the IVC, another chamber of liquid ⁴He (1 K pot) is fed by the outer ⁴He bath and can either be vented to atmosphere or connected to an external pump. Pumping the 1K pot reduces its temperature to ~1.2 K via evaporative cooling, allowing the gaseous ³He in the reservoir (sorb) to be condensed and collected in the ³He pot (when the sorb is heated to ~40 K). A charcoal sorption pump (SP) integrated within the ³He reservoir is turned on to further cool the ³He

pot down to the base temperature², forming the coldest section in the cryostat. The pumping speed of this SP is controlled by setting its temperature between approximately 4 K and 40 K. Once the base temperature is achieved, the SP heater is turned off, leaving the system in this temperature for several days (via evaporation of liquid ³He). The evaporated ³He is collected by the charcoal. When the ³He pot is empty, the cryostat returns to 4 K. At this point, one can heat the SP as well as the external pump of 1 K pot, to recondense the ³He and go back to the base temperature. It is worth mentioning that for the experiments in this thesis, we keep our system at 4 K. We did not go to base temperature mainly because the effect we want to demonstrate is insensitive to the bath temperature, not to mention that the recondensation procedure is time consuming.

Laser beams are directed into the cryostat via a single mode fiber inside the fridge (Fig. 3.6). The fiber terminates at a collimator inside the IVC so the light goes to freespace and bounces off a 45 degree steering mirror. Another (non-adjustable) 45 degree mirror directs the light towards the cavity input mirror. The reflected light from the cavity tracks the input path and is measured later on the optical table.

Both the collimator and the steering mirror are mounted on a three-axis piezo-electricallycontrolled mount (manufactured by Janssen Precision Engineering), which offers the tunability to align the laser beam with the cavity for optimal mode matching. These motorized mounts from Janssen are vacuum- and cryogenic- compatible. A third mount is used to adjust the titanium bridge that holds the membrane, and thus allows for rather long distance translation³ of the membrane along the cavity axis, as well as orientation (tip/tilt) control of the membrane.

It is important to maintain a strong thermal link between the experimental apparatus (especially the copper plate supporting the membrane) and the cold plate of the cryostat, while keeping the membrane vibrationally isolated from the cryostat. For this we use a large number of thin, loosely grouped, gold plated OFHC wires to provide the thermal connection, as illustrated in Fig. 3.6b.

^{2.} The designed base temperature of our system is \sim 300 mK, while in practice we found it to be \sim 450 mK without laser beams in the cavity.

^{3.} A distance of ~ 2 mm, which is much larger than the ~ 200 nm translation allowed by the ring piezo.



Figure 3.6: Cryogenic platform inside IVC. a, Schematic. The collimator, angled mirrors, cavity, and membrane are all mounted on the suspended Titanium platform. Thin thermal wires link the cold plate of the cryostat to the membrane support, as well as other parts of the platform. Suspension springs are critically damped with copper fins nesting between NdFeB magnets. b. Photograph of the cryogenic stage.

Several vibration isolation techniques are incorporated to reduce the seismic and acoustic noise from the environment. First of all, the entire cryostat is suspended on a pneumatically-floated frame (Newport S-2000A-128), and is surrounded by removable walls made of plexiglass covered in soundproofing foam. Secondly, the cavity and the supporting hardware are built on a \sim 1 kg titanium platform⁴ inside the IVC, which is suspended on critically damped springs. Critical damping of the springs is realized via magnetic damping, where eddy currents are induced with copper fins on the platform and NdFeB magnets mounted to a fixed plate underneath.

^{4.} Titanium is selected for its low thermal contraction at cryogenic temperatures.

3.2 Measurement setup

I now review the optical setup and the electronic circuits for detection. Additional details regarding specific experimental protocols will be presented in the relevant chapters.

3.2.1 Optics

We use two physically distinct lasers to generate the control beam and the measurement beam (Fig. 3.7a,b), and work with two cavity modes that are separated by $2 \times FSR$ (~ 8 GHz). This avoids the undesirable beat notes the control and measurement beams. Meanwhile, the membrane is located such that frequency shifts of the two cavity modes with respect to the change in the membrane position are roughly equal (Fig. 3.7c), which enables the locking between two laser beams (explained further in the next section).



Figure 3.7: Optical circuits. a, Schematic. ISO stands for isolator, and IM represents intensity modulator. b, Control and Measurement beams in frequency space. c, Position-dependence of different cavity modes, for a membrane located near the beam waist within the cavity.

The lasers used in our experiment are Prometheus Nd:YAG lasers manufactured by Innolight (now Coherent). These laser sources generate ~1 W beams with $\lambda \approx 1064$ nm, and the frequencies can be adjusted either via tuning the laser temperature or using a piezo attached to the YAG crystal.

The former way provides a slow, coarse control over a range of ~60 GHz, and the latter gives a fast (kHz scale bandwidth control) with a range of a few hundred MHz. The lasers are low noise with 1 kHz bandwidth, and include a built-in active intensity stabilizer ("noise eater"). To further suppress the phase noise, each laser can pass through a narrow linewidth (~20 kHz) Fabry-Pérot cavity that acts as a filter. The design of such filter cavities is shown in Fig. 3.8a. For most of the measurements carried out in this dissertation, both filter cavities are bypassed (via the alternative optical path shown in Fig. 3.8b).



Figure 3.8: Schematic of the filter cavity. a, Design. The piezo actuator (PZT) is used to adjust the cavity length at a kHz rate. b, Optical and electrical circuits. BLP1.9+: Mini-Circuits low pass filter (DC–1.9 MHz). PI Controller LB1005: proportional-integral controller for cavity frequency locking.

A small fraction of each beam is sampled using a beamsplitter immediately after the laser output. The beat note of these two samples is measured with a high-speed photodiode, and is used to stabilize the relative frequency between the two lasers (frequency lock in Fig. 3.7a, explained in next section).

After passing through (or bypassing) the filter cavity, the measurement laser is split into two beams: a weak probe and a much stronger local oscillator (LO). The probe beam then passes through an electro-optic modulator (EOM), which applies ~15 MHz phase modulation to allow for Pound-Drever-Hall (PDH) locking to the cryogenic cavity (explained in App. A). An acousto-optic modulator (AOM) shifts the probe (and its sidebands added by the EOM) by ~80 MHz, bringing them away from the LO frequency and close to the resonance of a cavity mode.

The control laser also passes through an 80 MHz AOM (AOM2 in Fig. 3.7a), after which the laser frequency is close to the resonance of another cavity mode that is 2×FSR higher than the measurement cavity mode. By modulating the radio frequency (RF) input that drives this AOM, we are able to tune the detuning and power of the control laser beam dynamically, which, as we will see

in Ch. 4, is important for our experiment. In fact, if we drive the AOM with multiple RF tones near 80 MHz, the modulated laser beam will include multiple tones correspondingly.

Later on, a fiber-based polarizing beamsplitter combines the measurement and control laser beams. The combined beam then passes through a fiber-coupled AOM, which shifts all the tones in the beam by 200 MHz. The PDH feedback signal goes into the voltage-controlled oscillator (VCO) that controls this AOM, therefore every tone in the combined beam tracks the fluctuations of the cryogenic cavity.

After the 200 MHz AOM, 1% of the beam power is directed to a photodiode via a beamsplitter for total laser power monitoring, while the rest of the beam is directed to the cryostat via a fiber-coupled circulator. The reflected light from the cryogenic cavity is redirected to another photodiode by this circulator, for the heterodyne measurement (explained in App. B).

3.2.2 Electronics



Figure 3.9: Schematic of electric circuits. DG1022: signal generator manufactured by Rigol. LSG-121: signal generator manufactured by Vaunix. ZI HF2: lock-in amplifier manufactured by Zurich Instrument, with six demodulation channels, two input channels and two output channels (only one channel shown here).

The reflected light from the cavity lands on an InGaAs photodiode (PDA10CF) with 150 MHz bandwidth. As illustrated in Fig. 3.9, the generated photocurrent goes to a bias tee and is separated into DC and RF components. The DC component, being proportional to the total beam power, is connected with an oscilloscope to monitor the reflection dip. The RF component, which includes beat notes between various pairs of laser tones, is split for both laser frequency locking and heterodyne signal processing. Therefore, the most relevant beatnotes are (i) the ~15 MHz beating between the probe and its (EOM controlled) sideband, and (ii) the ~80 MHz beating between the LO and the membrane added side bands on the probe. Note the RF component frequency is cut off at the
bandwidth of the photodiode, and is referred to below as the RF signal.

Along the feedback path, the RF signal is filtered by a low-pass filter at 21 MHz, and then mixes with a ~15 MHz reference signal (which also drives the measurement beam EOM). The down-mixed signal is filtered and amplified to form the PDH error signal.

Along the heterodyne measurement path, the RF signal is filtered by a band-pass filter centered at 80 MHz, and then mixes with a reference signal at 100 MHz. The down-mixed signal near 20 MHz is filtered and measured by a lock-in amplifier (Zurich Instrument HF2)⁵. Finally the demodulated signal is recorded for data processing.

3.3 Laser frequency locking

As we have seen in Ch. 1, detuning between the control laser frequency and its addressed cavity mode frequency is one of the important parameters that determine the behavior of an optomechanical system. Meanwhile, the probe beam (generated by the measurement laser) frequency should also track some cavity mode for the heterodyne measurement. However in practice, all frequencies may wander due to temperature fluctuations of the environment⁶. Therefore we employ locking techniques to stabilize the relative frequencies between the control/measurement lasers and the corresponding cavity modes.

Conceptually, the idea is to generate some error signal that varies linearly with frequency offset, and then apply an appropriate feedback to either the laser or the cavity. Here we implement a nested locking system that involves up to 5 interdependent feedback circuits, using proportional-integral (PI) controllers (New Focus LB1005) to create feedback signals. Specifically, we lock the probe beam on resonance with some cavity mode, and lock the control beam to the probe beam⁷. We can also lock the filter cavities if they are being used.

The probe beam frequency is locked to the cryogenic cavity via PDH technique (see App. A for how the error signal is generated). We implement the feedback in two ways. Firstly, the output of

^{5.} The reference signal is necessary since our HF2 can measure signal only up to 50 MHz.

^{6.} In addition, laser frequencies can drift due to mechanical imperfections and laser gain dynamics, while cavity mode frequencies can vary due to changes in membrane position.

^{7.} In principle, the two lasers can be locked separately to their corresponding cavity modes. The reason we choose to lock one beam to the other is to simplify the feedback circuits.



Figure 3.10: Schematic of laser frequency lockings. a, Measurement laser locking. b, Control laser locking. RS: Rohde&Schwarz SMB100A RF generator.

PI controller No. 1 is filtered and applied to the voltage-controlled oscillator (VCO) that drives the 200 MHz AOM (AOM3 in Fig. 3.7a), forming a fast feedback with bandwidth up to 5 kHz. The frequency of this AOM can only be varied between \sim 190 and 210 MHz, so in order to keep the driving frequency centered within this range, we use a second feedback channel. As shown in Fig. 3.10a, PI controller No. 2 takes the output of the PI controller No. 1 as its error signal and sends its feedback to the measurement laser piezo. This feedback is low-passed at 2 Hz, and is intended to handle any long-term drift in laser or cavity frequency, such that the driving tone of AOM3 is maintained \sim 200 MHz.

The control beam frequency is stabilized to the probe beam frequency via a slightly different circuit (Fig. 3.10b), similar to the scheme described in Ref. [145]. As mentioned in the previous section, the ~8 GHz beatnote between these two beams is captured by a high-speed photodiode. To stabilize the frequencies between the two lasers, we first bring this beatnote down to near DC by mixing the photocurrent with a reference tone (generated by Rohde&Schwarz SMB100A) at $\omega_{ref} \simeq 2 \times FSR$.⁸ Then we split the output of the mixer, send one arm through a 1.9 MHz low-pass filter, and recombine the two arms with a second mixer. The signal in the filtered arm experiences a steep, linear frequency-dependent phase shift whenever the frequency difference between two laser beams is near $\omega_{ref} \pm 1.9$ MHz. The resulting output of the second mixer is a voltage proportional to the original laser frequency difference, thus is used as the error signal. We send this error signal to PI controller No. 3 and apply the feedback to control laser piezo.

When filter cavities are in use, we need to keep their frequencies on resonance with the corre-

^{8.} Note that the ~16 GHz component is well suppressed due to the small bandwidth of the mixer.

sponding lasers. As shown in Fig. 3.8b, PDH technique is employed with the feedback applied to the piezo inside the filter cavity. Meanwhile, a feedforward signal split from the driving tone of AOM3 is applied to the measurement laser's filter cavity piezo, to further stabilize the locking.

In the end, we want to point out that in order to make our locking scheme actually work, the membrane should be located at a spot in the cryogenic cavity where the (absolute) slope of the cavity resonance frequency as a function of the membrane position is maximized and is approximately the same for both cavity modes (shown in Fig. 3.7c). Such locating yields a large optomechanical interaction, and makes the detuning of the control laser easily inferred. This condition is satisfied by initializing the membrane position (see in App. B for details).



Transient optomechanical nonreciprocity

We have seen in Ch. 1 that an EP in a non-Hermitian system reflects a non-trivial topology in the eigenvalues' dependence upon the system's parameters. In the simplest case (e.g., the 2×2 matrix H' in Sec. 1.3), the eigenvalue surfaces of the system's Hamiltonian (or dynamical matrix) possess the same topology as the Riemann sheets of the complex square root function. If the system is initialized with one normal mode, and its parameters then vary slowly in a closed loop that encircles an EP, the remaining excitation may transfer to the other normal mode that is involved in the EP [83, 146, 147]. Moreover, as we have shown in Ch. 1 and Ch. 2, the non-Hermitian dynamics during such operation ensures that the state (energy) transfer is nonreciprocal, with respect to the choice of the initial excitation as well as the sense of the control loop [98–100].

The nonreciprocity associated with this energy transfer arises from the interplay between the modes' coupling to each other and to their dissipative environment [124]. For a cavity optomechanical system, the couplings and dampings of mechanical modes can be controlled via the laser excitation in the cavity. As discussed in Ch. 2, we have demonstrated the nonreciprocal energy transfer between a pair of nearly-degenerate mechanical modes, within our MIM setup where a control laser tone can couple both mechanical modes and has *in situ* tunability [141].

In this chapter, we remove the constraint in the mechanical modes' near degeneracy, and extend such nonreciprcoal energy transfer to between any pair of mechanical modes. This is achieved by introducing an additional laser tone, which bridges the frequency gap between two well-separated mechanical modes and produces an EP in an appropriately defined rotating frame. This EP, referred to as a virtual exceptional point (VEP), offers the same nonreciprocity in energy transfer as the conventional one [142].

The chapter is organized as follows. I will start with a theoretical derivation of our two-tone scheme in Sec. 4.1, and then discuss the experimental details in Sec. 4.2 and the results in Sec. 4.3.

4.1 Theoretical derivation

We first build some intuitive understanding of the two-tone scheme. As illustrated in Fig. 4.1a, the cavity is driven by two laser tones at detunings $\Delta_1 \approx -\omega_1$ and $\Delta_2 \approx -\omega_2$, where ω_i is the natural frequency of mechanical mode i (i = 1, 2). Qualitatively, such arrangement ensures the motion of mechanical mode 1 (2) produces a sideband from tone 1 (2) that is approximately on resonance with the cavity, and the beating between this sideband and tone 2 (1) causes the intracavity light intensity to oscillate at a frequency close to ω_1 (ω_2). Therefore, any near-resonant motion in one mechanical mode will exert a near-resonant driving force on the other, regardless of the difference in their natural frequencies (Fig4.1b).



Figure 4.1: Bichromatic control beam induced coupling. a, Spectrum of the bichromatic control beam. Two laser tones (light green and dark green) drive a single cavity mode to generate the coupling between two mechanical modes. $\Delta_{1,2}$ are the laser detunings and $\omega_{1,2}$ are natural frequencies of the mechanical modes. δ measures the overlap between the anti-Stokes sidebands. b, A microscopic picture of the coupling induced by the two control laser tones. The solid horizontal lines are states labelled by the number of phonons in each mechanical mode (n_1, n_2) and the number of cavity photons (n_c) . The hollow lines represent the process where tone 1(2) absorbs a phonon from mechanical mode 1(2) and creates a cavity photon, and then generates a phonon in mechanical mode 2(1) via tone 2(1). The dashed horizontal line is a virtual state through which the transfer process occurs. $\Omega_{1,2}$ are the absolute frequencies of the control laser tones.

4.1.1 Bichromatic light mediated mechanical coupling

We now provide a quantitative description of the effective coupling between two mechanical modes.

Consider a cavity optomechanical system, where the cavity is driven by two laser tones (red-detuned

with respect to a cavity mode at frequency ω_c), we write down the system's equations of motion:

$$\dot{a}(t) = -(\frac{\kappa}{2} + i\omega_c)a(t) - i\sum_{k=1}^2 g_k(c_k^*(t) + c_k(t))a(t) + \sqrt{\kappa_{\rm in}}a_{\rm in}(t)$$
(4.1)

$$\dot{c}_{k}(t) = -(\frac{\gamma_{k}}{2} + i\omega_{\mathrm{m},k})c_{k}(t) - ig_{k}|a(t)|^{2}$$
(4.2)

where Δ_n is the detuning of tone *n* (*n* = 1, 2), and the input field of the cavity is defined as

$$a_{\rm in}(t) = e^{-i\omega_{\rm c}t} \sum_{n=1}^{2} a_{{\rm in},n} e^{-i\Delta_n t} = e^{-i\omega_{\rm c}t} \sum_{n=1}^{2} \sqrt{\frac{P_n}{\hbar(\omega_{\rm c} + \Delta_n)}} e^{-i\Delta_n t}$$
(4.3)

Similar to the monochromatic driving case, we can linearize the equations above by setting the cavity field to fluctuate around some mean value, and neglect high-order terms of the fluctuation. Specifically, we look for the solution with form $a(t) = (\bar{\alpha} + d)e^{-i\omega_c t}$, where $\bar{\alpha} = \sum_{n=1}^{2} \alpha_n e^{-i\Delta_n t}$ and $\alpha_n = \frac{\sqrt{\kappa_{in}}a_{in,n}}{\kappa/2-i\Delta_n}$. With proper substitutions, we have:

$$\dot{d}(t) = -\frac{\kappa}{2}d(t) - i\sum_{k=1}^{2} g_k \bar{\alpha}(c_k^*(t) + c_k(t))$$
(4.4)

$$\dot{c}_{k}(t) = -(\frac{\gamma_{k}}{2} + i\omega_{\mathrm{m},k})c_{k}(t) - ig_{k}(\bar{\alpha}^{*}d(t) + \bar{\alpha}d^{*}(t))$$
(4.5)

where we have ignored the constant forces as well as the drivings at frequency $|\Delta_1 - \Delta_2|^1$.

We want to eliminate d(t) in Eq. (4.5) in order to derive the effective coupling between the two mechanical modes. Rewriting the equations in the Fourier domain, we have:

$$-i\omega d[\omega] = -\frac{\kappa}{2} d[\omega] - i\sum_{k=1}^{2}\sum_{n=1}^{2}g_k\alpha_n(c_k^*[\omega - \Delta_n] + c_k[\omega - \Delta_n])$$
(4.6)

$$-i\omega c_k[\omega] = -(\frac{\gamma_k}{2} + i\omega_{m,k})c_k[\omega] - ig_k \sum_{n=1}^2 (\alpha_n^* d[\omega + \Delta_n] + \alpha_n d^*[\omega - \Delta_n])$$
(4.7)

We point out the RHS of Eq. (4.6) and Eq. (4.7) contains Fourier component other than frequency at ω because $\bar{\alpha}$ is time dependent. Recall the mechanical susceptibility function $\chi_{m,k}(x) = (\gamma_k/2 - i(\omega - \omega_{m,k}))^{-1}$ (see in Ch. 2), and define a quantity $\chi(x) = (\frac{\kappa}{2} - ix)^{-1}$ in analogous to the optical

^{1.} We assume $|\Delta_1 - \Delta_2| \ll \omega_{m,1}, \omega_{m,2}$ throughout this dissertation.

susceptibility, we have:

$$d[\omega + \Delta_n] = -i\chi(\omega + \Delta_n) \sum_{l=1}^2 \sum_{m=1}^2 \alpha_m g_l(c_l^*[\omega + \Delta_{nm}] + c_l[\omega + \Delta_{nm}])$$
(4.8)

$$d^*[\omega - \Delta_m] = i\chi(\omega - \Delta_m) \sum_{l=1}^2 \sum_{n=1}^2 \alpha_n^* g_l(c_l^*[\omega + \Delta_{nm}] + c_l[\omega + \Delta_{nm}])$$
(4.9)

$$c_k[\omega] = \chi_{\mathrm{m},k}(\omega) \left(-ig_k \sum_{n=1}^2 (\alpha_n^* d[\omega + \Delta_n] + \alpha_n d^*[\omega - \Delta_n])\right)$$
(4.10)

where $\Delta_{nm} = \Delta_n - \Delta_m$ denotes the relative detuning between the two tones. Eliminating *d* and *d*^{*} in Eq. (4.10), we get:

$$c_{k}[\omega] = -i\chi_{m,k}(\omega)g_{k}\sum_{n=1}^{2}\alpha_{n}^{*}d[\omega + \Delta_{n}] - i\chi_{m,k}(\omega)g_{k}\sum_{n=1}^{2}\alpha_{n}d^{*}[\omega - \Delta_{n}]$$

$$= -i\chi_{m,k}(\omega)g_{k}\sum_{n=1}^{2}\alpha_{n}^{*}d[\omega + \Delta_{n}] - i\chi_{m,k}(\omega)g_{k}\sum_{m=1}^{2}\alpha_{m}d^{*}[\omega - \Delta_{m}]$$

$$= -\chi_{m,k}(\omega)g_{k}\sum_{n=1}^{2}\alpha_{n}^{*}\chi(\omega + \Delta_{n})\sum_{l=1}^{2}\sum_{m=1}^{2}\alpha_{m}g_{l}(c_{l}^{*}[\omega + \Delta_{nm}] + c_{l}[\omega + \Delta_{nm}])$$

$$+\chi_{m,k}(\omega)g_{k}\sum_{m=1}^{2}\alpha_{m}\chi(\omega - \Delta_{m})\sum_{l=1}^{2}\sum_{n=1}^{2}\alpha_{n}^{*}g_{l}(c_{l}^{*}[\omega + \Delta_{nm}] + c_{l}[\omega + \Delta_{nm}])$$

$$= \chi_{m,k}(\omega)g_{k}\cdot\sum_{l=1}^{2}\sum_{m=1}^{2}\sum_{n=1}^{2}g_{l}\alpha_{m}\alpha_{n}^{*}(\chi(\omega - \Delta_{m}) - \chi(\omega + \Delta_{n}))$$

$$\times (c_{l}^{*}[\omega + \Delta_{nm}] + c_{l}[\omega + \Delta_{nm}])$$
(4.11)

Note that the subscriptions *m* and *n* under summations in Eq. (4.11) are interchangeable. To simplify Eq. (4.11), we rely on the fact that the quality factors of the mechanical modes are high², and the detunings of the two laser tones are chosen such that $\Delta_{12} \approx \omega_{m,2} - \omega_{m,1}$. Looking at the RHS of Eq. (4.11), we first ignore c_k^* terms as they correspond to the contributions at $\approx -\omega_{m,k}$. For the remaining terms, when l = k, the main contributions come from terms with m = n, which correspond to the motion in the mode itself; when l > k (l < k), we can just keep the near-resonant m > n (m < n)

^{2.} In the high-Q limit, $|c_k[\omega]|$ reaches maximum near $\omega_{m,k}$ (specific location depends on the optical spring effect), and becomes negligible when $|\omega - \omega_{m,k}| \gg \gamma_k^f (\gamma_k^f$ represents the light-mediated mechanical linewidth).

cross terms. The approximate equations of motion in the frequency domain are then:

$$\chi_{1}(\omega)^{-1}c_{1}[\omega] = g_{1}^{2} \sum_{n=1}^{2} |\alpha_{n}|^{2} (\chi(\omega - \Delta_{n}) - \chi(\omega + \Delta_{n}))c_{1}[\omega]$$

$$+ g_{1}g_{2}\alpha_{1}^{*}\alpha_{2}(\chi(\omega - \Delta_{2}) - \chi(\omega + \Delta_{1}))c_{2}[\omega + \Delta_{12}]$$
(4.12)

$$\chi_{2}(\omega)^{-1}c_{2}[\omega] = g_{2}^{2} \sum_{n=1}^{2} |\alpha_{n}|^{2} (\chi(\omega - \Delta_{n}) - \chi(\omega + \Delta_{n}))c_{2}[\omega]$$

$$+ g_{1}g_{2}\alpha_{2}^{*}\alpha_{1}(\chi(\omega - \Delta_{1}) - \chi(\omega + \Delta_{2}))c_{2}[\omega + \Delta_{21}]$$
(4.13)

from which we can set the argument of χ in front of c_k to $\omega_{m,k}$, and convert the equations back into the time domain:

$$\dot{c}_1(t) = -(\frac{\gamma_1}{2} + i\omega_{\rm m,1} + i\sigma_{11})c_1(t) - i\sigma_{12}e^{i\Delta_{12}t}c_2(t)$$
(4.14)

$$\dot{c}_2(t) = -(\frac{\gamma_2}{2} + i\omega_{\rm m,2} + i\sigma_{22})c_1(t) - i\sigma_{21}e^{i\Delta_{21}t}c_2(t)$$
(4.15)

where

$$\sigma_{kk} = ig_k^2 \sum_{n=1}^2 |\alpha_n|^2 (\chi(\omega_{\mathrm{m},k} - \Delta_n) - \chi(\omega_{\mathrm{m},k} + \Delta_n))$$
(4.16)

$$\sigma_{mn} = ig_1g_2\alpha_m^*\alpha_n(\chi(\omega_{m,m} - \Delta_n) - \chi(\omega_{m,m} + \Delta_m))$$
(4.17)

If we define the mechanical amplitude vector $c(t) = (c_1(t), c_2(t))^T$, and introduce a timedependent force vector $f(t) = (f_1(t), f_2(t))^T$ that drives the mechanical modes (e.g., random thermal fluctuations), we finally have the equations of motion in matrix form:

$$i\dot{\boldsymbol{c}}(t) = \boldsymbol{D}(t)\boldsymbol{c}(t) + \boldsymbol{f}(t) \tag{4.18}$$

where the time-dependent dynamical matrix D(t) is expressed as:

$$D(t) = \begin{pmatrix} \omega_{m,1} - i\gamma_1/2 + \sigma_{11} & \sigma_{12}e^{i\Delta_{12}t} \\ \sigma_{21}e^{i\Delta_{21}t} & \omega_{m,2} - i\gamma_2/2 + \sigma_{22} \end{pmatrix}$$
(4.19)

We can see that the off-diagonal terms of D(t) are comparable with the dynamical backaction terms

in the diagonal, indicating an effective coupling between the two mechanical modes.

4.1.2 Eigenvalue spectrum in a rotating frame

The light-mediated frequencies and linewidths of the two mechanical modes can be theoretically solved by finding the real and imaginary parts in the eigenvalues of the dynamical matrix defined in Eq. (4.19). In the lab frame, since the frequencies of the two mechanical modes are well-separated from each other, there is no degeneracy in the eigenvalue spectrum of D(t). However, we will now show that we can enter a rotating frame such that the "rotated" eigenvalues coalesce.

Consider a rotated amplitude vector $c^{r}(t)$, defined as:

$$\boldsymbol{c}^{\mathrm{r}}(t) = U(t)\boldsymbol{c}(t) = \begin{pmatrix} e^{-i\frac{\Delta_{12}}{2}t} & 0\\ 0 & e^{i\frac{\Delta_{12}}{2}t} \end{pmatrix} \boldsymbol{c}(t) = \begin{pmatrix} e^{-i\frac{\Delta_{12}}{2}t}c_1(t)\\ e^{i\frac{\Delta_{12}}{2}t}c_2(t) \end{pmatrix}$$
(4.20)

We can write down the equation of motion that describes the evolution of $c^{r}(t)$:

$$i\dot{\boldsymbol{c}}^{\mathrm{r}}(t) = i\dot{U}(t)\boldsymbol{c}(t) + iU(t)\dot{\boldsymbol{c}}(t) = \begin{pmatrix} \frac{\Delta_{12}}{2} & 0\\ 0 & -\frac{\Delta_{12}}{2} \end{pmatrix} \boldsymbol{c}^{\mathrm{r}}(t) + i\begin{pmatrix} e^{-i\frac{\Delta_{12}}{2}t} & 0\\ 0 & e^{i\frac{\Delta_{12}}{2}t} \end{pmatrix} \dot{\boldsymbol{c}}(t)$$

$$= \frac{\Delta_{12}}{2}\sigma_{z}\boldsymbol{c}^{\mathrm{r}}(t) + U\boldsymbol{D}(t)\boldsymbol{U}^{-1}\boldsymbol{c}^{\mathrm{r}}(t) + U\boldsymbol{f}(t) = \boldsymbol{D}^{\mathrm{rot}}\boldsymbol{c}^{\mathrm{r}}(t) + \boldsymbol{f}^{\mathrm{r}}(t)$$
(4.21)

where we have introduced the rotated force vector $f^{r}(t) = U(t)f(t)$ and dynamical matrix:

$$D^{\text{rot}} = \frac{\Delta_{12}}{2}\sigma_z + UDU^{-1} = \begin{pmatrix} \omega_{\text{m},1} + \frac{\Delta_{12}}{2} - i\frac{\gamma_1}{2} + \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \omega_{\text{m},2} - \frac{\Delta_{12}}{2} - i\frac{\gamma_2}{2} + \sigma_{22} \end{pmatrix}$$
(4.22)

Note that D^{rot} is time-independent, and since $\Delta_{12} \approx \omega_{m,2} - \omega_{m,1}$, it can be considered as the dynamical matrix of two nearly-degenerate mechanical modes at frequencies $\omega_{m,1} + \Delta_{12}/2$ and $\omega_{m,2} - \Delta_{12}/2$. Therefore, with appropriate choice of laser parameters (i.e., power and detuning), one may establish a degeneracy in eigenvalues of D^{rot} .

This degeneracy is referred to as a VEP in our system, because at this point, the corresponding eigenvalues of the original dynamical matrix D(t) remain distinct. To understand the behavior of the system near such a VEP, we first look at how the eigenvalues of D^{rot} are measured. Experimentally, the eigenvalues of a time-independent dynamical matrix can be extracted from a driven response

measurement. In this measurement, a driving force at a certain frequency is applied to the system, and the system's response at this frequency is recorded. The susceptibility function of the system can then be measured by sweeping the drive over a frequency band.

Specifically, suppose we drive a mechanical mode k (to some amplitude $c_k(t)$) with a force $f_k(t)$. In the Fourier domain, there exists an effective susceptibility function $\chi_{m,k}^{\text{eff}}(\omega)$ that satisfies $c_k[\omega] = \chi_{m,k}^{\text{eff}}(\omega) f_k[\omega]$.³ To derive the form of $\chi_{m,k}^{\text{eff}}(\omega)$, we apply the Fourier transform to Eq. (4.21)⁴ and get:

$$\boldsymbol{c}^{\mathrm{r}}[\omega] = (\omega I_{2\times 2} - D^{\mathrm{rot}})^{-1} \boldsymbol{f}^{\mathrm{r}}[\omega]$$
(4.23)

Solving for motions in the lab frame yields:

$$c_{1}[\omega] = c_{1}^{r}[\omega + \Delta_{12}/2] = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot ((\omega + \Delta_{12}/2)I_{2\times 2} - D^{rot})^{-1} \cdot \begin{pmatrix} f_{1}[\omega] \\ f_{2}[\omega + \Delta_{12}] \end{pmatrix}$$

$$c_{2}[\omega] = c_{2}^{r}[\omega - \Delta_{12}/2] = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot ((\omega - \Delta_{12}/2)I_{2\times 2} - D^{rot})^{-1} \cdot \begin{pmatrix} f_{1}[\omega - \Delta_{12}] \\ f_{2}[\omega] \end{pmatrix}$$
(4.24)

where we have used $f_1^r[\omega] = f_1[\omega - \Delta_{12}/2]$ and $f_2^r[\omega] = f_2[\omega + \Delta_{12}/2]$. More explicitly,

$$c_{1}[\omega] = \frac{(\omega - \omega_{m,2} - \sigma_{22} + \Delta_{12} + i\gamma_{2}/2)f_{1}[\omega] + \sigma_{12}f_{2}[\omega + \Delta_{12}]}{(\omega - \omega_{m,1} - \sigma_{11} + i\gamma_{1}/2)(\omega - \omega_{m,2} - \sigma_{22} + \Delta_{12} + i\gamma_{2}/2) - \sigma_{12}\sigma_{21}}$$
(4.25)

$$c_{2}[\omega] = \frac{(\omega - \omega_{m,1} - \sigma_{11} - \Delta_{12} + i\gamma_{1}/2)f_{2}[\omega] + \sigma_{21}f_{1}[\omega - \Delta_{12}]}{(\omega - \omega_{m,1} - \sigma_{11} - \Delta_{12} + i\gamma_{1}/2)(\omega - \omega_{m,2} - \sigma_{22} + i\gamma_{2}/2) - \sigma_{12}\sigma_{21}}$$
(4.26)

In practice, $f_k[\omega]$ is centered at $\omega_{m,k}$ (k = 1, 2), and the response of mode k is maximized near $\omega_{m,k}$, so the "cross driving" terms (f_2 in Eq. (4.25) and f_1 in Eq. (4.26)) can be ignored. We finally have:

$$\chi_{m,1}^{\text{eff}}(\omega) = \frac{(\omega - \omega_{m,2} - \sigma_{22} + \Delta_{12} + i\gamma_2/2)}{(\omega - \omega_{m,1} - \sigma_{11} + i\gamma_1/2)(\omega - \omega_{m,2} - \sigma_{22} + \Delta_{12} + i\gamma_2/2) - \sigma_{12}\sigma_{21}} \chi_{m,2}^{\text{eff}}(\omega) = \frac{(\omega - \omega_{m,1} - \sigma_{11} - \Delta_{12} + i\gamma_1/2)}{(\omega - \omega_{m,1} - \sigma_{11} - \Delta_{12} + i\gamma_1/2)(\omega - \omega_{m,2} - \sigma_{22} + i\gamma_2/2) - \sigma_{12}\sigma_{21}}$$
(4.27)

One may notice that the denominators in Eq. (4.27) are characteristic polynomials of $D^{\text{rot}} \pm$

^{3.} We have seen in Ch. 2 that $\chi_{m,k}^{\text{eff}}(\omega) = \chi_{m,k}(\omega)$ for a bare damped harmonic oscillator.

^{4.} We do not apply the Fourier transform to Eq. (4.18), because the time-dependent D(t) leads to multiple frequency components of c, which is less convenient to be dealt with.

 $\frac{\Delta_{12}}{2}I_{2\times 2}$. When the eigenvalues of D^{rot} are non-degenerate, the effective susceptibility can be written as a sum of two complex Lorentzians. For example,

$$\chi_{m,1}^{\text{eff}}(\omega) = \frac{h_1}{\omega - \tilde{\omega}_+} + \frac{h_2}{\omega - \tilde{\omega}_-}$$
(4.28)

where $\tilde{\omega}_{\pm}$ are complex roots of the denominator in $\chi_{m,1}^{\text{eff}}(\omega)$ (i.e., eigenvalues of D^{rot} downward shifted⁵ by $\Delta_{12}/2$), and the complex height h_1 (h_2) is defined as

$$h_{1} = \frac{1}{2} + \frac{\omega_{m,2} - \omega_{m,1} - \Delta_{12} + \sigma_{22} - \sigma_{11} - i(\gamma_{2} - \gamma_{1})/2}{2\sqrt{(\omega_{m,2} - \omega_{m,1} - \Delta_{12} + \sigma_{22} - \sigma_{11} - i(\gamma_{2} - \gamma_{1})/2)^{2} + 4\sigma_{12}\sigma_{21}}}$$

$$h_{2} = \frac{1}{2} - \frac{\omega_{m,2} - \omega_{m,1} - \Delta_{12} + \sigma_{22} - \sigma_{11} - i(\gamma_{2} - \gamma_{1})/2}{2\sqrt{(\omega_{m,2} - \omega_{m,1} - \Delta_{12} + \sigma_{22} - \sigma_{11} - i(\gamma_{2} - \gamma_{1})/2)^{2} + 4\sigma_{12}\sigma_{21}}}$$

$$(4.29)$$

Eq. (4.28) indicates that the mechanical driven response near $\omega_{m,1}$ consists of two Lorentzian shapes centered at Re[$\tilde{\omega}_{\pm}$] with linewidths $|2\text{Im}[\tilde{\omega}_{\pm}]|$, respectively. A similar treatment of $\chi_{m,2}^{\text{eff}}(\omega)$ shows that a driven measurement around $\omega_{m,2}$ will have Lorentzians centered at Re $\tilde{\omega}_{\pm} + \Delta_{12}$ with the same corresponding linewidths (though the heights are different). As a result, if we shift the driven response near $\omega_{m,1}$ ($\omega_{m,2}$) by $\Delta_{12}/2$ upwardly (downwardly), the centers of the Lorentzians in the shifted responses will overlap with each other (as illustrated in Fig. 4.2).



Figure 4.2: Sample of shifted driven response result. Red (Blue) dots are the measured driven response near frequency $\omega_{m,1}$ ($\omega_{m,2}$), with the plotting shifted up (down) by $\Delta_{12}/2$. The vertical dashed line corresponds to $\Re \tilde{\omega}_+ + \Delta_{12}/2$, where an overlap is observed. All numerical values are to be specified in Sec. 4.2.

A special case occurs when $\tilde{\omega}_+ = \tilde{\omega}_-$, corresponding to a degeneracy in eigenvalues of D^{rot} (or a

^{5.} We assume $\omega_{m,1} < \omega_{m,2}$ throughout the remainder of this dissertation.

VEP of the system in the lab frame). Eq. (4.28) is now replaced with:

$$\chi_{m,1}^{\text{eff}}(\omega) = \frac{h_{\text{el}}}{\omega - \tilde{\omega}_{\text{e}}} + \frac{h_{\text{e2}}}{(\omega - \tilde{\omega}_{\text{e}})^2}$$
(4.30)

where $\tilde{\omega}_{e}$ is the degenerate eigenvalue, and h_{ei} are complex heights which can be determined by Eq. (4.27). According to Eq. (4.30), the driven response appears to be single-peaked, while the shape is not a simple Lorentzian. Although mathematically distinguishable, fluctuations in parameters of any experimental setup will prevent a system from settling onto this special point, so we will model and fit our data according to Eq. (4.28).

4.2 Experimental implementation

In this experiment, we focus on the {2, 1} and {2, 2} modes of the membrane, with natural frequencies $\omega_{m,1} = 2\pi \times 557.473 \text{ kHz}, \ \omega_{m,2} = 2\pi \times 705.164 \text{ kHz}$ and bare linewidths $\gamma_1 = 2\pi \times 0.39 \text{ Hz},$ $\gamma_2 = 2\pi \times 0.38 \text{ Hz}$. The optomechanical coupling rates are $g_1 = 2\pi \times 2.11 \text{ Hz}$ and $g_2 = 2\pi \times 2.12 \text{ Hz}.$

As discussed in Sec. 1.3, two linearly independent parameters are sufficient to demonstrate the existence of an EP (or a VEP). For a two-tone driven optomechanical system, four parameters (i.e., the power and detuning of each tone) can be varied to tune the system's dynamical matrix. To reduce the dimension of the control parameter space, we set the two tones to be equal in power $(P \equiv P_1 = P_2)$, and fix $\Delta_{12} = \omega_{m,2} - \omega_{m,1} + \delta$ to define a common detuning $\Delta \equiv \Delta_1 + \omega_{m,1} = \Delta_2 + \omega_{m,2}$. The parameter δ determines the overall detuning of the coupling shown in Fig. 4.1. For example, if $\delta = 0$, the resonant motion of one mode drives the other mode exactly on resonance. We set $\delta = 2\pi \times 100$ Hz, which is comparable to the optically damped mechanical linewidths.

Eigenvalues of the rotated dynamical matrix D^{rot} can now be solved for a given (Δ , P). Numerical calculation shows that a degeneracy is achieved at $\Delta_{\text{VEP}} = -2\pi \times 15$ kHz and $P_{\text{VEP}} = 4.7 \ \mu\text{W}$ (illustrated in Fig. 4.3). To measure these eigenvalues experimentally, recall that the driven response of a mechanical mode can be used to extract the resonance frequency and the linewidth at a fixed point in (Δ , P) space. We may vary both control parameters over a grid spanning a wide range in this space to trace out the eigenvalue spectrum (details follow later in this section). After that, we employ loops encircling the VEP to observe the nonreciprocal energy transfer.



Figure 4.3: Eigenvalue spectrum of D^{rot} near a VEP, calculated numerically. The frequencies plotted are numbers relative to the average mechanical frequeny $(\omega_{m,1} + \omega_{m,2})/2$.

4.2.1 Measurement of mechanical driven responses

We perform two driven measurements (near 557 kHz and 705 kHz) at each chosen point in the parameter space. As outlined in Ch. 3 and the previous paragraphs, the control laser generates two tones, described by (Δ , *P*), and the measurement laser provides the heterodyne signal. The drive of the mechanical modes are created optically, by applying an amplitude modulation at frequency ω_{drive} to the RF signal generator that drives the measurement laser AOM (i.e., AOM1 in Fig. 4.5). The beat note between the 1st order modulation sideband and the measurement beam is at ω_{drive} , and therefore applies an oscillating radiation pressure force to the membrane.⁶. The magnitude and phase of the response at ω_{drive} is recorded by the HF2, where an internal narrow-bandwidth filter (≤ 1 Hz) is applied to the signal. To have the response as a function of ω_{drive} , this driving frequency (controlled by the HF2) is swept over a frequency band.

An example of such a driven measurement is shown in Fig. 4.4, where the frequencies are shifted for the sake of better visualization. Specifically, the driven response near 557 kHz is shifted upward by $\Delta_{12}/2 \approx 75$ kHz, and the driven response near 705 kHz is shifted downward by the same amount. We emphasize that we have entered the rotating frame defined in Eq. (4.20) via such post-processing of the response signal. This can be appreciated by recalling the mapping between motions in the lab frame and in the rotating frame (see in Eq. (4.24)), as well as the fact that measured response signal of mode *k* is proportional to the amplitude c_k .

^{6.} Alternatively, one can drive the mechanical modes via the piezo attached to the membrane support. But such drives seem to excite resonances somewhere in the membrane support structure, which introduces giant additional noise.



Figure 4.4: Sample driven responses of mechanical modes. a, Shifted data of the measurement near $\omega_{m,1} \approx 557$ kHz. Both real and imaginary parts are plotted as functions of the driving frequency. Double-Lorentzian lineshape of the amplitude indicates the coupling to the other mode. The coupling can also be noticed in the plot of real vs. imaginary quadratures, where the trace looks like two half circles sitting on top of each other. b, Shifted data of the measurement near $\omega_{m,2} \approx 705$ kHz. Solid black lines are results from the fitting described in the main text.

To extract the eigenvalues of D^{rot} from these measurements, we fit the two (shifted) data sets simultaneously according to the theoretical model Eq. (4.28). Explicitly, the fitting functions are:

$$b_{1} + \frac{h_{1}}{\omega - \omega_{+} + i\frac{\gamma_{+}}{2}} + \frac{h_{2}}{\omega - \omega_{-} + i\frac{\gamma_{-}}{2}} \text{ for response near } \omega_{m,1}$$

$$b_{2} + \frac{h_{3}}{\omega - \omega_{+} + i\frac{\gamma_{+}}{2}} + \frac{h_{4}}{\omega - \omega_{-} + i\frac{\gamma_{-}}{2}} \text{ for response near } \omega_{m,2}$$

$$(4.31)$$

with b_i , h_i , ω_{\pm} and γ_{\pm} as fitting parameters. b_1 (b_2) is a complex noise background, h_1 (h_2 , h_3 , h_4) is a complex height of a Lorentzian, and ω_{\pm} (γ_{\pm}) are the centers (linewidths) of the two Lorentzians. Since our theoretical model suggests that the complex heights may not be independent from each other (e.g., $(h_1 + h_2)/(h_3 + h_4)$ =const), there is redundancy in the above fitting parameters. However, as we only care about ω_{\pm} and γ_{\pm} , it is convenient to leave all the h_i as free parameters.

4.2.2 Measurement of nonreciprocal energy transfers

We can study nonreciprocal topological energy transfers once the existence of an VEP is established. To measure the transfer efficiency (defined below) experimentally, we first initialize the system with one mechanical mode (by driving it to several orders of magnitude larger than its thermal motion), then turn on the two-tone control laser and vary the parameters (Δ and *P*) to form a closed loop in (Δ , *P*) space that encircles the VEP, after which the energy in both modes is measured.

The initial excitation of a chosen mechanical mode is achieved by applying a sinusoidal drive to the membrane piezo⁷. We generate two tones in the control laser by driving an AOM (i.e., AOM2 in Fig. 4.5) with two frequencies (separated by Δ_{12}) near 80 kHz. This RF driving signal comes from mixing a 100 MHz oscillator with an output from the lock-in amplifier (HF2) near 20 MHz. The HF2 uses two internal oscillators for both creating the output and demodulating⁸ the heterodyne signal.



Figure 4.5: Schematic of electric circuits for VEP experiment. AWG: arbitrary waveform generator (we use Rigol DG1022 signal generators in this experiment).

In order to vary Δ and *P* dynamically, we apply a sequence of voltages to both the amplitude and frequency modulation ports of the 100 MHz local oscillator (which up-mixes the HF2 output

^{7.} We do not use optical drive as we need a much larger excitation to compete with the overall damping during the experiment.

^{8.} The bandwidth δf for these demodulators should be set carefully. On the one hand, the minimum time for a channel to respond to changes in the signal is proportional to $1/\delta f$, so we want δf to be large. On the other hand, a very large δf may include the motion from the other mode. In this experiment, we set $\delta f = 50$ Hz, such that the motion in each mode is recorded separately. As a result, we have a response time of $\delta t = 20$ ms that is relevant in the data processing.

and drives the control laser AOM). For simplicity reasons, we perform rectangular loops defined by four points (Δ_{\min} , P_{\min}), (Δ_{\min} , P_{\max}), (Δ_{\max} , P_{\max}), (Δ_{\max} , P_{\min}), returning to (Δ_{\min} , P_{\min}) after a total time τ , as illustrated in Fig. 4.6. The voltage ramps that control a loop are provided by Rigol DG1022 signal generators (AWG2 and AWG3 in Fig. 4.5).



Figure 4.6: Sample measurement of the nonreciprocal energy transfer by encircling a VEP. **a**, Schematic of a clockwise loop with $\Delta_{\min} = -2\pi \times 604$ kHz, $\Delta_{\max} = 2\pi \times 374$ kHz, $P_{\min} = 0.08 \mu$ W, $P_{\max} = 8.3 \mu$ W. **b**, Sample lock-in amplitudes with an initialization in high frequency mode ($\omega_{m,2} \approx 705$ kHz) and the loop defined in **a**. The loop time $\tau \approx 27$ ms. We observe energy transfer as the low frequency mode is excited after the loop. **c**, Sample measurement with the same loop as in **b**, but initialized with low frequency mode ($\omega_{m,1} \approx 557$ kHz). After the loop, the other mode is not excited. The gray lines in **b**, **c** indicate a period during which we have no accurate information on the eigenmode amplitudes/energies. Eigenmode energies are converted from the voltage signal (explained later in the main text).

To synchronize the mode initialization, the control loop and the signal recording, a square wave from another Rigol DG1022 signal generator (AWG1 in Fig. 4.5) serves as a clock for the whole measurement process. Part of this square wave provides the control voltage for an RF switch, which turns on the membrane piezo drive when the control signal is low. The square wave is also fed to AWG2 and AWG3 as an external trigger. When it goes high, the mechanical drive is cut off, whereas AWG2 and AWG3 will output the user-defined voltage ramps (which generate a parameter loop). Finally, this waveform is recorded by an auxiliary input of the HF2, for the purpose of data processing. In practice, the square wave should be at the low level long enough for the mechanical mode to be excited to the desired amplitude, while it should stay in the high level during the loop, the lock-in response (20 ms), and the mechanical modes' relaxation. As illustrated in Fig. 4.7, the whole measurement is repeated several times to increase the signal to noise ratio of the recorded voltage.



Figure 4.7: Schematic of the measurement protocol. From top to bottom: the trigger signal for synchronization, the sinusoidal drive for mechanical mode initialization, the power modulation profile, the detuning modulation profile. From left to right: four cycles of the measurement, and a zoom-in illustration on one cycle.

To process the collected data, we first use the sync signal to align all the individual measurements (shown in Fig. 4.8). Next, we convert the voltage signals into the mechanical energy in each mode (measured by the number of phonons n_k). The conversion factor can be derived via calculating the heterodyne spectrum of a mechanical oscillator equilibrated within a thermal bath [148], such that the phonon number in mode *k* converted from the red (blue) sideband of the measurement beam is:

$$n_k = \frac{V_{\rm r(b)}\hbar\omega_c}{G_{\rm r(b)}\sigma^2\beta^2\kappa_{\rm in}g_0^2|\chi(\mp\omega_{\rm m,k})|P_{\rm lo}P_{\rm meas}}$$
(4.32)

where $V_{\rm r(b)}$ is the voltage signal, $G_{\rm r} = 5.548 \times 10^{-16}$ V/Hz ($G_{\rm b} = 5.533 \times 10^{-16}$ V/Hz) is the electrical gain of the photodiode at frequency $\omega_{\rm if} \mp \omega_{\rm m,k}$ (with $\omega_{\rm if} = 80.5$ MHz being the separation between the LO beam and the measurement beam), $\beta = 0.877$ is the dimensionless photon coupling rate, $\sigma = 0.359$ is the dimensionless detection efficiency [123]. We may then calculate the energy in each mechanical mode by $E_k = n_k \hbar \omega_{\rm m,k}$.



Figure 4.8: Alignment of individual transfer measurements. The faint red (blue) lines represent individual energy transfer measurement results. The solid black lines are the averaged results.

We use the averaged, converted data to quantify the amount and efficiency of the energy transfer during the loop. Suppose the loop is employed during $[0, \tau]$. The information we need is $E_k(\tau)$ for k = 1, 2. As shown in Fig. 4.9, we fit the energy over a wide range of $t > \tau$, where τ represents the loop time. Note that the fitting window is offset from τ by $\delta t = 20$ ms intentionally. This is because demodulators of the HF2 only capture the motion at frequencies set by the internal oscillators, while the instantaneous eigenmode frequencies change as the parameters vary, which makes the data recorded during the control loop meaningless. The data within $[\tau, \tau + \delta t]$ is thus not accurate, as it contains signals during the loop (given the limited bandwidth of the demodulators).

Since both modes relax to thermal equilibrium after the loop, $E_k(t)$ $(t > \tau)$ can be described by an exponential decay function. Once the fitting is completed, we extrapolate the result backward to get $E_k(\tau)$ (black dots in Fig. 4.9). The efficiency of the energy transfer can be defined as $\eta_f = E_f(\tau)/(E_1(\tau) + E_2(\tau))$, where f = 1 (2) if mode 2 (1) is excited initially. The transfer is nonreciprocal when $\eta_1 \neq \eta_2$.

One may also calculate the magnitude of elements of the propagation matrix $U(\tau)$ for such an energy transfer process ($c(\tau) = U(\tau)c(0)$, see in Ch. 2 for results with nearly-degenerate modes). For example, consider a clockwise loop with initialized energy in mode *i*, the propagation matrix element can be expressed as:

$$|U_{\cup,ji}(\tau)| = \frac{|c_j(\tau)|}{|c_i(\tau)|} = \frac{|\sqrt{E_j(\tau)}|}{|\sqrt{E_i(\tau)}|}$$
(4.33)



Figure 4.9: Sample fitting of the mechanical motions. The loop (data) is described in Fig. 4.6a (b). Red (blue) lines are the averaged signal. Black lines are fittings to the exponential function. The energy in each mode right after the loop is indicated by the black dot.

4.3 Data analysis

With details of the measurement explained in the previous section, we now focus on the postprocessed data of this experiment.

4.3.1 Eigenvalue spectrum near VEP

Fig. 4.10 shows the eigenvalues of D^{rot} in the vicinity of an VEP. In the (Δ , P) parameter space, both the real and imaginary parts of the eigenvalue spectrum exhibit a sharp feature as they approach the VEP (at $\Delta \approx -15$ kHz, $P \approx 4.7 \mu$ W). Our measurement result agrees well with the theoretical predictions (smooth gray sheets in Fig. 4.10a, b and e), which are solved by fitting the data to a full optomechanical model. In fact, this global fit to all the data points (extracted from fitting the driven response measurements) can be used as a good estimation of some system parameters, such as the natural mechanical frequencies $\omega_{m,1(2)}$, linewidths $\gamma_{1(2)}$, and the optomechanical couplings $g_{1(2)}$.



Figure 4.10: Eigenvalue spectrum near a VEP. **a** (**b**), The frequencies (linewidths) extracted from a set of driven response measurements, plotted in the rotating frame. The smooth gray sheets are least-squares fit, with $(\omega_{m,1}, \omega_{m,2}, g_1, g_2)$ as fitting parameters. **c**, **d**, Fitting residuals against Δ and *P*. **e**, Eigenvalue differences calculated from data in **a** and **b**.

4.3.2 Nonreciprocity as a function of time

To examine the nonreciprocity in the energy transfer processes, we measure the transfer efficiency for loops encircling the VEP in both the CCW and CW senses, with the initial excitation in either mode 1 or mode 2. For each of the four possible combinations, we perform loops with the same shape defined in Fig. 4.6a, and the loop duration τ changes from 1 ms to 40 ms.

Fig. 4.11 shows the energy transfer efficiency η all these cases. As $\tau \to 0$, the transfer efficiency goes to zero, which is expected for a diabatically (suddenly) perturbed system. In the adiabatic limit $(\tau \to \infty \text{ mathematically, and } \tau > 10 \text{ ms in practice})$, we can see $\eta \to 1$ if the loop is CCW (CW) and the system is initialized with mode 1 (2), while $\eta \to 0$ if the loop is CW (CCW) and the system is initialized with mode 1 (2). The nonreciprocity observed here reflects the fact that during an adiabatic evolution, the system tries to follow the topological structure of the eigenvalue spectrum, while the difference between the damping rate in the two modes tends to leave the system in the less-damped mode [98].



Figure 4.11: Energy transfer efficiency as a function of the loop duration. **a**, Clockwise loops. Red (blue) points mean the system is initialized in mode 1 (2). **b**, The same as **a** except the loops are counterclockwise. The solid black lines in both **a** and **b** are results from numerical simulations of the evolution described by Eq. 4.18, with the system parameters given in the previous section.

CHAPTER 2

Static optomechanical nonreciprocity

In previous chapters, transient nonreciprocity has been demonstrated within our optomechanical setup by driving the optical cavity with two control laser tones, whose powers and detunings are time-dependent. More precisely, to realize nonreciprocal phononic energy transfer, the (common) power and detuning should be varied in an adiabatic way such that an EP or VEP is encircled [141, 142]. By contrast, in this chapter I will describe a scheme for achieving nonreciprocity with stationary modulation and continuous operation. Moreover, since our experiment directly measures the device's internal degrees of freedom, it differs from typical experiments on a nonreciprocal device (in photonics as well as other fields) that measure the scattering matrix that describes propagating waves landing on and emanating from the device. This feature allows us to control the states of a system of resonators via the nonreciprocal interactions between them. We illustrate this by using the nonreciprocity to control the direction of heat flow between two modes of the membrane, thereby realizing a qualitatively new way to laser-cool the mechanical resonators.

I will begin with some insight on the two-tone scheme discussed in Ch. 4 that motivates our current scheme, and then present a theoretical description of the current scheme (Sec. 5.1). The experimental implementation, which is just slightly modified from the previous chapter, will follow in Sec. 5.2. This chapter ends with a discussion on the measurement results (Sec. 5.3).

5.1 Theoretical derivation

Recall that in Ch. 4, near-resonant coupling between two (non-degenerate) mechanical modes can be induced by modulating the dynamical backaction at a frequency near $\delta \omega_m = \omega_{m,2} - \omega_{m,1}$. As shown in Fig. 5.1a, such modulation arises from the intracavity beat note between a pair of cavity drive with detunings $\Delta_1 = -\omega_{m,1} + \Delta_l + \zeta$, $\Delta_2 = -\omega_{m,2} + \Delta_l$ (where ζ is comparable to the mechanical linewidths). In this arrangement, a photon can scatter from one drive tone to the other by transferring a phonon between the modes. This process (illustrated by the light and dark red arrows in Fig. 5.1b) occurs via a virtual state in which the photon is at an anti-stokes sideband of the drive tones.



Figure 5.1: Four-tone scheme for nonreciprocal coupling between two mechanical modes. **a**, The frequency domain illustration. The gray curve is the cavity lineshape. The thin dash line is the cavity resonance, and the thick dash lines are negative detunings equal to the mechanical mode frequencies $\omega_{m,1(2)}$. The colored arrows are control tones with detunings (with respect to the cavity resonance) Δ_i (i = 1, 2, 3, 4), whose motional sidebands (that dominate the phonon transfer process) are represented by colored Lorentzians. The horizontal axis shows detuning from cavity resonance. **b**, The energy domain illustration. The solid horizontal lines are states labelled by the number of phonons in each mode (n_1, n_2) and the number of cavity photons (n_c). The absolute frequency of the *i*th control tone is Ω_i . The dashed horizontal lines are virtual states through which the transfer process occurs. The cavity linewidth is indicated by the grey shading.

We emphasize two crucial features of the modulation induced phonon transfer. Firstly, the transfer amplitude is proportional to the complex-valued cavity susceptibility $\chi(\Delta_l)$ (where $\chi(\omega) = 1/(\kappa - i\omega)$ is introduced in Ch. 4) regardless of the direction of transfer, and thus has both a dissipative and a coherent character. Secondly, the phase of the intracavity beat note appears explicitly in the transfer coefficient. While these features alone do not result in nonreciprocal energy transfer (e.g., the beat note phase can be gauged away), interference between two such processes can break reciprocity. As we will see below, the additional phonon transfer process is achieved by incorporating a second pair of drive tones (thus a second beat note) into the cavity. Moreover, the interference is controlled by the relative phase between the two beat notes and therefore can not be gauged away.

5.1.1 Nonreciprocal coupling in a four-tone scheme

Suppose the cavity is driven by four red-detuned control laser tones, with detunings Δ_n , powers P_n , and phases ϕ_n (n = 1, 2, 3, 4). The detunings of the four tones are chosen to provide two beat notes that each induce near-resonant coupling between the two mechanical modes (i.e., $\Delta_1 - \Delta_2 = \Delta_3 - \Delta_4 \approx \delta \omega_m$). Hence there are two distinct copies of the phonon transfer process. The four detunings are also chosen such that the dominant mechanical sideband in each transfer process has a distinct detuning $\Delta_{l(u)}$, where $\Delta_l = \Delta_1 + \omega_{m,1} - \zeta = \Delta_2 + \omega_{m,2}$ and $\Delta_u = \Delta_3 + \omega_{m,1} - \zeta = \Delta_4 + \omega_{m,2}$.

We can describe this system via the standard linearized optomechanical equations of motion for one cavity mode and two mechanical modes. Specifically, with the cavity drive:

$$a_{\rm in} = \sum_{n=1}^{4} \sqrt{\frac{P_n}{\hbar\Omega_n}} e^{-i\omega_{\rm c}t + \Delta_n t + \phi_n} = e^{-i\omega_{\rm c}t} \sum_{n=1}^{4} \sqrt{\frac{P_n}{\hbar(\omega_{\rm c} + \Delta_n)}} e^{-i(\Delta_n t + \phi_n)}$$
(5.1)

we linearize the optical field around a coherent amplitude $\tilde{\alpha}$ by setting $a = e^{-i\omega_{c}t}(\tilde{\alpha} + d)$, such that $\tilde{\alpha} = \sum_{n=1}^{4} \alpha_{n} e^{-i(\Delta_{n}t + \phi_{n})}$ where $\alpha_{n} = \sqrt{\frac{\kappa_{in}P_{n}}{\hbar(\omega_{c} + \Delta_{n})}} \frac{1}{\kappa/2 - i\Delta_{n}}$, and *d* is the fluctuation of the cavity mode. The system's equations of motion (in a frame that oscillates at ω_{c}) can be written as:

$$\dot{d}(t) = -\frac{\kappa}{2}d(t) - i\sum_{k=1}^{2} g_k \tilde{\alpha}(c_k^*(t) + c_k(t))$$
(5.2)

$$\dot{c}_k(t) = -(\frac{\gamma_k}{2} + i\omega_{\mathrm{m},k})c_k(t) - ig_k(\tilde{\alpha}^*d(t) + \tilde{\alpha}d^*(t))$$
(5.3)

which are identical to Eq. (4.4) and Eq. (4.5) except $\bar{\alpha} \to \tilde{\alpha}$. Therefore we may follow the steps in Ch. 4 to adiabatically eliminate the cavity field, and derive the dynamical matrix:

$$D(t) = \begin{pmatrix} \omega_{m,1} - i\gamma_1/2 + \sigma_{11} & \sigma_{12}e^{i\Delta t} \\ \sigma_{21}e^{-i\Delta t} & \omega_{m,2} - i\gamma_2/2 + \sigma_{22} \end{pmatrix}$$
(5.4)

where the diagonal components represent the usual single-tone dynamical backaction:

$$\sigma_{kk} = \sum_{n=1}^{4} i g_n^2 |\alpha_n|^2 (\chi(\omega_{\mathrm{m},k} - \Delta_n) - \chi(\omega_{\mathrm{m},k} + \Delta_n))$$
(5.5)

and the off-diagonal components describe the coupling between the two mechanical modes mediated

by the intracavity beat notes:

$$\sigma_{12} = \sum_{n=1}^{2} i g_1 g_2 \alpha_{2n-1}^* \alpha_{2n} e^{i(\phi_{2n-1} - \phi_{2n})} (\chi(\omega_{m,1} - \Delta_{2n}) - \chi(\omega_{m,1} + \Delta_{2n-1}))$$

$$\sigma_{21} = \sum_{n=1}^{2} i g_1 g_2 \alpha_{2n-1} \alpha_{2n}^* e^{-i(\phi_{2n-1} - \phi_{2n})} (\chi(\omega_{m,2} - \Delta_{2n-1}) - \chi(\omega_{m,2} + \Delta_{2n}))$$
(5.6)

Note that $\Delta = \Delta_1 - \Delta_2 = \Delta_3 - \Delta_4$.

To understand how strong nonreciprocal coupling can be induced in the system, consider

$$|D_{12}(t)| = |\sigma_{12}| \approx g e^{i(\theta_l + \phi_{12})} + h e^{i(\theta_u + \phi_{34})}$$

$$|D_{21}(t)| = |\sigma_{21}| \approx g e^{i(\theta_l - \phi_{12})} + h e^{i(\theta_u - \phi_{34})}$$
(5.7)

where we have defined the phase differences $\phi_{12} = \phi_1 - \phi_2$ and $\phi_{34} = \phi_3 - \phi_4$. The real coefficients are $g \approx g_1 g_2 |\alpha_1^* \alpha_2 \chi(\Delta_l)|$, $h \approx g_1 g_2 |\alpha_3^* \alpha_4 \chi(\Delta_u)|$, where small terms in g and h that are due to non-resonant mechanical sidebands are ignored. (For clarity, these terms are ignored just for the discussion here, and will be included in the analysis and fits presented later in this chapter.) The isolation between the two mechanical modes can be achieved by choosing parameters to make $|D_{12}| \ll |D_{21}|$ or $|D_{12}| \gg |D_{21}|$. To achieve this, we first set P_n and Δ_n such that $|g| \approx |h|^1$. We then adjust the phases ϕ_{12} and ϕ_{34} to ensure that one off-diagonal element of D(t) almost vanishes while the other does not. This is illustrated in Fig. 5.2, where D_{12} and D_{21} are plotted as functions of $\phi = \phi_{12} - \phi_{34}$. We can see that $|D_{12}| \gg |D_{21}|$ when $\phi \approx -\pi/2$, so the energy is allowed to flow from mode 1 to mode 2 but not vice versa. The situation is reversed for $\phi \approx +\pi/2$. By contrast, $\phi \approx 0$ ($\phi \approx \pm \pi$) gives $|D_{12}| \approx |D_{21}|$, so the energy transport is almost reciprocal. Notably, the tunability between isolation, reciprocity and reverse isolation is realized by only varying the ϕ , while keeping P_n and Δ_n fixed.

^{1.} In this chapter, we focus on the same two mechanical modes as in Ch. 4. $|g| \approx |h|$ can be satisfied by choosing $P_n = 5 \ \mu W$, $\Delta_l = -2\pi \times 60 \ \text{kHz}$, $\Delta_u = 2\pi \times 150 \ \text{kHz}$, $\zeta = 2\pi \times 100 \ \text{Hz}$.



Figure 5.2: The off-diagonal elements of the dynamical matrix. The real and imaginary parts of D_{12} and D_{21} (in units of Hz) are plotted parametrically versus the relative phase ϕ between the two beat notes. The powers and detunings of control laser tones are given in the main text. The color scale encodes the value of ϕ . The grey (brown) arrow indicates the evolution of D_{12} (D_{21}) as ϕ increases from 0.

5.1.2 Asymmetric cooling in a common thermal bath

Let us apply the four-tone scheme to modify the thermal fluctuations of mechanical modes. In general, a mechanical resonator is in the thermally steady state when detailed balance between its modes and the environment is achieved. To describe the steady-state thermal fluctuations of our system, we note that both mechanical modes couple to a thermal bath ($T_{\text{bath}} = 4.2 \text{ K}$) and to the cavity field (a bath with effective $T_C \approx 0$ as $\hbar \omega_c \gg k_B T_{\text{bath}}$). In the absence of couplings between the modes, if we turn on a red-detuned monochromatic control beam, each mode will equilibrate to a temperature $T_a = (\gamma_a^f)^{-1} \gamma_a T_B$ (a = 1, 2), where γ_a^f is the single tone optical damping rate². This reduction of T_a with respect to T_{bath} is known as the effect of cold damping or laser cooling [113]. If the modes interact with each other (e.g., via the aforementioned two-tone or four-tone scheme),

^{2.} We assume $\gamma_a^f \gg \gamma_a$ throughout this chapter.

thermal phonons may transport between the modes and lead to new steady-state temperatures that have more complicated expressions (which we discuss below). In the case where the energy transport is reciprocal ($|D_{12}| = |D_{21}|$), thermal phonons are exchanged between the modes, which tends to bring T_1 and T_2 closer to each other. When the energy transport is unidirectional, by constrast, thermal phonons are emitted from the isolated mode into the other mode, but not *vice versa*. This results in cooling of the isolated mode (and heating of the other mode), even if the former is the colder of the two in the non-interacting case.

To quantitatively describe this isolation-based cooling, we introduce the concept of effective temperature T_a^{eff} for a mode *a*, which is associated to the effective phonon number n_a^{eff} via $T_a^{\text{eff}} = n_a^{\text{eff}} \hbar \omega_{\text{m},a}/k_{\text{B}}^3$. The effective phonon number in mode *a* can be expressed as an integral of the membrane spectrum $S_a^{c^*c}[\omega]$ over the frequency space:

$$n_a^{\text{eff}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_a^{c^*c}[\omega] d\omega$$
(5.8)

where by definition:

$$S_a^{c^*c}[\omega] \equiv \langle c_a^*[\omega]c_a[-\omega] \rangle = \int_{-\infty}^{\infty} e^{i\omega\tau} \langle c^*(t)c_a(t+\tau) \rangle d\tau$$
(5.9)

Note that the second equivalence results from the Wiener-Khinchin theorem [149].

We can derive $S_a^{c^*c}[\omega]$ by solving the equations of motion in the Fourier domain⁴:

$$c_1[\omega] = \tilde{\chi}_1(\omega) (\sigma_{12} c_2[\omega + \delta \omega_m] + \sqrt{\gamma_1} \eta_1[\omega])$$
(5.10)

$$c_2[\omega] = \tilde{\chi}_2(\omega) (\sigma_{21} c_1[\omega - \delta \omega_m] + \sqrt{\gamma_2} \eta_2[\omega])$$
(5.11)

where we have introduced the optically mediated susceptibilities $\tilde{\chi}_a(\omega) = (\gamma_a/2 - i(\omega - \omega_{m,a}) + \sigma_{aa})^{-1} = (\tilde{\gamma}_a/2 - i(\omega - \tilde{\omega}_a))^{-1}$ for a = 1, 2. (The real numbers $\tilde{\omega}_a$, $\tilde{\gamma}_a$ represent the modulated mechanical frequencies and dampings). The Fourier component $\eta_a[\omega]$ of the Langevin force $\eta_a(t)$

^{3.} The discussion here is just pedagogical, since the effective "mode frequency" is not well defined in a coupled-mode system. The general form of the effective temperature will be addressed later.

^{4.} Suppose the mechanical modes are only driven by the thermal noise (Langevin force), and we ignore the noise in control laser tones.

(defined in Ch. 2) satisfies:

$$\langle \eta_a[\omega]\eta_a^*[-\omega]\rangle = \langle \eta_a^*[\omega]\eta_a[-\omega]\rangle = n_a^{\text{th}}$$

$$\langle \eta_a^*[\omega]\eta_b[-\omega]\rangle = \langle \eta_a[\omega]\eta_b^*[-\omega]\rangle = 0$$

$$(5.12)$$

Write down explicitly the solutions to Eq. (5.10) and Eq. (5.11):

$$c_{1}[\omega] = \tilde{\chi}_{1}(\omega) \frac{\sqrt{\gamma_{1}}\eta_{1}[\omega] + \sigma_{12}\tilde{\chi}_{2}(\omega + \delta\omega_{m})\sqrt{\gamma_{2}}\eta_{2}[\omega + \delta\omega_{m}]}{1 - \tilde{\chi}_{1}(\omega)\tilde{\chi}_{2}(\omega + \delta\omega_{m})\sigma_{12}\sigma_{21}}$$

$$c_{2}[\omega] = \tilde{\chi}_{2}(\omega) \frac{\sqrt{\gamma_{2}}\eta_{2}[\omega] + \sigma_{21}\tilde{\chi}_{1}(\omega - \delta\omega_{m})\sqrt{\gamma_{1}}\eta_{1}[\omega + \delta\omega_{m}]}{1 - \tilde{\chi}_{2}(\omega)\tilde{\chi}_{1}(\omega - \delta\omega_{m})\sigma_{12}\sigma_{21}}$$
(5.13)

and treat c_a (a = 1, 2) as sums of complex Lorentzians similar to Eq. (4.28). Then substitute c_a and c_a^* in Eq. (5.9) with the solutions above, after the integration:

$$n_{1}^{\text{eff}} = c_{11}n_{1}^{\text{th}} + |\sigma_{12}|^{2}c_{12}n_{2}^{\text{th}}$$

$$n_{2}^{\text{eff}} = |\sigma_{12}|^{2}c_{21}n_{1}^{\text{th}} + c_{22}n_{2}^{\text{th}}$$
(5.14)

where the coefficients can be found in App. A. The ratio between effective temperatures of the two modes is therefore:

$$\frac{T_2^{\text{eff}}}{T_1^{\text{eff}}} \propto \frac{n_2^{\text{eff}}}{n_1^{\text{eff}}} = \frac{c_{11}n_1^{\text{th}} + |\sigma_{12}|^2 c_{12}n_2^{\text{th}}}{c_{22}n_2^{\text{th}} + |\sigma_{21}|^2 c_{21}n_1^{\text{th}}}$$
(5.15)

We can see from Eq. (5.15) that the ratio depends on the off-diagonal terms in the Hamiltonian, and thus is a function of ϕ . We will demonstrate this experimentally in the next section.

Experimentally, the thermal motion of a mechanical mode *a* is encoded in the heterodyne signal, which can be converted into the displacement spectral density S_a^{xx} . Since $S_a^{xx}[\omega] = 2x_{zpf}^2 S_a^{c^*c}[\omega]$ in the classical regime, we can calibrate the effective phonon number by integrating the measured spectral density over a wide range in the frequency domain (usually set by the bandwidth of the demodulators in the HF2).

To end this section, we now provide a more rigorous definition of T^{eff} . In general, the effective temperature of a stationary system (not necessarily in thermal equilibrium) can be defined at each frequency, by considering the ratio between the fluctuation and dissipation at that frequency [149]. Letting S_a^{xx} denote the displacement spectral density and $\chi_a^{xx}[\omega]$ be the full mechanical susceptibility

(i.e., the retarded x - x Green's function) of the mode we are studying, we have:

$$\coth \frac{\hbar\omega}{2k_{\rm B}T_a(\omega)} \equiv \frac{S_a^{xx}[\omega] + S_a^{xx}[-\omega]}{-2\hbar \mathrm{Im}\chi_a^{xx}[\omega]}$$
(5.16)

If the effective temperatures are much larger than the frequencies of interest, i.e., $\hbar\omega_a \ll 2k_BT_a(\omega)$, this relation becomes

$$\bar{S}_{a}^{xx}[\omega] \equiv \frac{S_{a}^{xx}[\omega] + S_{a}^{xx}[-\omega]}{2} = -2k_{\rm B}\frac{T_{a}(\omega){\rm Im}\chi_{a}^{xx}[\omega]}{\omega}$$
(5.17)

which we integrate to get

$$\langle x_a^2 \rangle = \int_{-\infty}^{\infty} \bar{S}_a^{xx}[\omega] d\omega = \int_{-\infty}^{\infty} -2k_{\rm B} \frac{T_a(\omega) {\rm Im} \chi_a^{xx}[\omega]}{\omega} d\omega$$
(5.18)

As we can see from Eq. (5.18), the position fluctuation $\langle x_a^2 \rangle$ is a weighted integral of $T_a(\omega)$, and thus can be used to define a single effective temperature $\overline{T} = k_a^{\text{eff}} \langle x_a^2 \rangle / k_{\text{B}}$, where k_a^{eff} is the effective spring constant:

$$k_a^{\text{eff}} = \left(\int_{-\infty}^{\infty} \frac{-2\text{Im}\chi_a^{xx}[\omega]}{\omega} d\omega\right)^{-1} = -(\text{Im}\chi_a^{xx}[\omega=0])^{-1}$$
(5.19)

Note that we have used the Kramers-Kronig relation in the last equivalence. This definition of effective temperature does not require the mechanical mode to have a Lorentzian resonance, but if the system truly equilibrated at a physical temperature T_0 , then it follows (via the fluctuation-dissipation theorem) that $T_a(\omega) = T_0$, irrespective of the frequency ω and the shape of χ_a^{xx} . For the mechanical oscillator in our system, the bare spring constant $k_a^0 = m\omega_{m,a}^2$, and we can verify that $k_a^{\text{eff}} \approx k_a^0$, therefore $\overline{T}_a = T_a^{\text{eff}}$. We will denote the effective temperature of mode *a* by T_a in the remainder of this chapter.

5.2 Experimental implementation

The experimental setup in Ch. 4 can be easily modified to allow for a second pair of control laser tones. We achieve this by including two additional oscillators of the HF2 into the output that (is first mixed up to a frequency near 80 MHz and then) drives the control laser AOM, as illustrated in Fig. 5.3.



Figure 5.3: Electric circuit schematic for the four-tone scheme. To generate a control laser profile with four separate tones, four channels of the HF2 are used to output signals that will be mixed up to drive the control laser AOM (AOM2 in the schematic). The power, detuning, and phase of each control laser tone can be adjusted on the HF2 control panel via the amplitude, frequency and phaseshift of the corresponding oscillator. The two channels left are used to demodulate the heterodyne signal, monitoring the mechanical motions.

Since the nonreciprocity is tuned via adjusting the relative phase between the control laser tones, we remove the FM/AM drive that is triggered by the square wave. Otherwise, the setup is similar to that of Ch. 4: separate demodulation channels near $\omega_{m,1}$ and $\omega_{m,2}$ are used to record the amplitude (energy) of each mechanical mode, while the membrane can be driven optically by a signal at frequency ω_{drive} that modulates the measurement laser AOM.

5.3 **Results and discussion**

Without loss of generality, we set $\phi_1 = \phi_2 = \phi_3 = 0$, thus the control phase $\phi = \phi_4$ can be adjusted with one knob. Bare frequencies and linewidths, as well as the optomechanical coupling rates of the two mechanical modes have been presented in Ch. 4. The settings on the HF2 are chosen such that $P_n \approx 5 \ \mu\text{W}, \ \Delta_l = -2\pi \times 60 \ \text{kHz}, \ \Delta_u = -2\pi \times 150 \ \text{kHz}, \ \zeta = 2\pi \times 100 \ \text{Hz}.$

5.3.1 Measurement of nonreciprocal energy transfers

To measure the amount of energy transfer from one mode to the other, say $1 \rightarrow 2$ ($2 \rightarrow 1$), we first excite mode 1(2) to a large amplitude, by applying a sinusoidal drive to the membrane piezo

at $\omega_{m,1}$ ($\omega_{m,2}$). Then at t = 0, we stop the piezo drive, and turn on the control tones to couple the two mechanical modes for a duration $[0, \tau]$ (shown in Fig. 5.4). The energy transfer between these modes occurs during this duration. At $t = \tau$, the control tones are turned off, and then the motion of each mode decays freely to their thermal equilibrium (i.e., for $t > \tau$).



Figure 5.4: Measured mechanical energy in each mode as a function of time. **a**, The system is initialized in mode 1. The control laser tones are on only during the grey region $0 \le t \le \tau = 3$ ms with $\phi = \pi/2$. Data for $t > \tau + \delta f = 23$ ms is fitted to a decaying exponential (black curves) and this fit is extrapolated to t = 3 ms to find $E_1(\tau)$ and $E_2(\tau)$, which represent the energies in each mode at the end of the control pulse (black dots at t = 3 ms). **b**, Same control beam but system initialized in mode 2.

The amount of energy transfer that has taken place can be calculated from the amplitude of the mechanical motions at $t = \tau$. Due to the limited response time ($\delta f = 20 \text{ ms}$) of the HF2 demodulators and the thermal noise in the mechanical motions, this information is gained by first fitting the modes' decay in a large time window ($t > \tau + \delta f$) to an exponential function, and then extrapolating the fit to $t = \tau$ ms. For the purpose of a better signal-to-noise ratio, the initialization and measurement processes are repeated many times (see in Ch. 4 for details on data averaging).

5.3.2 Demonstration of tunability and robustness

To demonstrate the tunability of the nonreciprocity, we measure the transfer of energy between the two modes for various choices of ϕ . We then calibrate the energy transmission coefficients $T_{\uparrow}(\phi) \equiv E_2^{\phi}(\tau)/E_1(0)$ and $T_{\downarrow}(\phi) \equiv E_1^{\phi}(\tau)/E_2(0)$, which correspond to transfer from mode 1 to mode 2 and *vice versa*, and plot them as functions of ϕ in Fig. 5.5.



Figure 5.5: Energy transmission coefficients plotted against the control phase ϕ . The error bars for the statistical uncertainties are smaller than the symbols. The solid lines are the numerical simulation from the model discussed in the previous section.

A common practice is to introduce the isolation ratio $I(\phi) \equiv T_{\uparrow}(\phi)/T_{\downarrow}(\phi)$, which is illustrated in Fig. 5.6. We observe that the maximum isolation is realized near $\phi = \pi/2$, where $I(\pi/2) \approx \max_{\phi} I(\phi) > 30$ dB. At this point, the energy transfer in the system is approximately unidirectional from mode 1 to mode 2. Strong nonreciprocal energy transfer in the opposite direction is reached near $\phi = -\pi/2$, where $I(-\pi/2) \approx \min_{\phi} I(\phi) < -25$ dB. The reciprocity is restored ($I(\phi) = 0$ dB, corresponding to $|H_{12}| = |H_{21}|$) when $\phi \approx 0$. The figure also shows that *I* can be tuned over the entire range by varying ϕ with all other parameters fixed. The solid lines in the figure are not fits but the predictions from numerically evolving the theoretical dynamical matrix D(t).⁵

^{5.} D(t) is no longer time dependent as we enter a proper rotating frame. The evolution can therefore be calculated by taking the matrix exponential of D.



Figure 5.6: Isolation as a function of ϕ . The values of *I* are extracted from the data in Fig. 5.5. The error bars for the statistical uncertainties are smaller than the symbols. The solid line is the theoretical prediction described in the main text.

We then consider the transmission coefficients $T_{\uparrow(\downarrow)}$ and the isolation ratio *I* as functions of the control tones' duration τ . Measurement result of the transmission coefficients at $\phi = 0, \pm \pi/2$ is shown in Fig. 5.7. Both T_{\uparrow} and T_{\downarrow} decrease with τ , owing primarily to the damping induced by the single-tone backaction since all four control tones are red-detuned from the cavity resonance.



Figure 5.7: The transmission coefficients as functions of the control tones' duration. The error bars for the statistical uncertainties are smaller than the symbols. The solid lines are the theoretical prediction described in the main text.

By contrast, $I(\phi)$ remains independent of τ , as illustrated in Fig. 5.8. Thus, we have demonstrated a robust, compact, stationary and tunable scheme for inducing nonreciprocity between phononic resonators.



Figure 5.8: The isolation ratio I as a function of the control tones? duration τ . The error bars for the statistical uncertainties are smaller than the symbols. The solid lines are the theoretical prediction described in the main text.

5.3.3 Realization of asymmetric cooling

The derivation in the previous section indicates that we can adjust the ratio between the mode temperatures by only varying ϕ . To demonstrate this nonreciprocal effect on the mode temperatures, we calibrate the effective temperature of each mode from the motional sideband of the corresponding mode. Fig. 5.9 shows three samples of the membrane's power spectral density (which is proportional to S^{xx}), plotted as a function of frequency. In comparison with the energy transfer measurement, no external drive is applied to the phonon modes, so we are simply recording the modes' Brownian motion. All parameters remain the same as for the energy transfer measurements, except that the control laser tone power is cut down to $P_n = 2.5 \mu$ W. We reduce the power simply to avoid the laser unlocking during the data taking process.



Figure 5.9: The power spectral density of the two modes' thermal motions. Similar to the driven measurement results in Ch. 4, the data have been offset horizontally such that the two modes (which oscillate near 557 kHz and 705 kHz) can be compared directly. From left to right, the three panels correspond to $\phi = -\pi/2$, 0 and $+\pi/2$. The solid black lines are fitting results with functions in form Eq. (5.20).

As discussed in Ch. 4, the energy spectral density can be converted from the recorded heterodyne signal with an appropriate scaling factor. The effective temperatures T_1 and T_2 are then determined from the area under the peaks in $S_{E,1}$ and $S_{E,2}$, which are shown in Fig. 5.9 for $\phi = 0, \pm \pi/2$. Note that the lineshape of the energy spectral density is not a single Lorentzian near the resonance of each mode. Such asymmetric lineshapes are commonly observed in nearly-degenerate modes of a system of damped harmonic oscillators [150, 151]; while in the present system the mechanical modes are non-degenerate, and the lineshapes reflect the interference between two paths via which a given mode is driven by the thermal bath. Mathematically, $S_{E,a}$ near $\omega_{m,a}$ (a = 1, 2) is a constant background plus the square modulus of the sum of two Lorentzians, and can be described by:

$$b_a + \left| \frac{u_\alpha}{\frac{\gamma_\alpha}{2} - i(\omega - \omega_\alpha)} + \frac{u_\beta}{\frac{\gamma_\beta}{2} - i(\omega - \omega_\beta)} \right|^2$$
(5.20)

where $\omega_{\alpha(\beta)}$, $\gamma_{\alpha(\beta)}$, and $u_{\alpha(\beta)}$ are the resonance frequency, linewidth and complex height of the Lorentzian peak α , respectively. Suppose for mechanical mode a, peak α corresponds to single Lorentzian lineshape when there is no intermode-coupling, then T_a is calibrated by the area of $S_{E,a}$ under this peak, which is $\pi |u_{\alpha}|^2 / \gamma_{\alpha}$.



Figure 5.10: The calibrated temperature for both modes as a function of ϕ . The solid lines are theoretical predictions. This data is used to calculate the normalized temperature difference shown in Fig. 5.11a.

Fig. 5.10 shows the measured T_1 and T_2 for a set of different ϕ s varying from $-\pi$ to π . To better visualize the effect of nonreciprocity on the mode temperatures, we define the normalized temperature difference

$$\Theta(\phi) = 1 - \frac{T_2(\phi)/T_1(\phi)}{\langle T_2/T_1 \rangle_{\phi}}$$
(5.21)

where $\langle ... \rangle_{\phi}$ denotes an average over ϕ . The merit of introducing this quantity is twofold. Firstly, the extreme values of $\Theta(\phi)$ indicates the maximum isolation between the modes. (Note the fact that $\Theta(\pm \pi/2)$ being close to extremum is consistent with the previous result on nonreciprocal energy transfers.) Secondly, changing the sign of Θ is equivalent to reversing the direction of heat flow between the modes. Specifically, heat is transported from the colder mode to the hotter mode in two scenarios: { $\langle T_2/T_1 \rangle_{\phi} < 1, \Theta > 0$ } and { $\langle T_2/T_1 \rangle_{\phi} > 1, \Theta < 0$ }. In the present setup we have $\langle T_2/T_1 \rangle_{\phi} = 1.79 > 1$ (calculated from the data in Fig. 5.10), therefore the colder 557 kHz mode is "cooled" by the hotter 705 kHz mode, as long as $\Theta < 0$.


Figure 5.11: Normal temperature difference as a function of ϕ , measured for four detuning offsets $\Delta_{\text{off}} = 0$, $-2\pi \times 100$ kHz, $-2\pi \times 200$ kHz, $-2\pi \times 300$ kHz.

The data in Fig. 5.10 is converted to Θ in Fig. 5.11a, where the solid line shows the theoretical Θ calculated from the optomechanical equations of motion. The agreement between the measured and predicted cooling extends over a wide range of parameters. For example, $\Theta(\phi)$ with different detuning offset (i.e., frequencies of all four control tones move together) Δ_{off} is illustrated in Fig. 5.11b–d. Note the parameters at $\Delta_{\text{off}} = 0$ are the same as those in the energy transfer measurement.

We emphasize that the data in each panel of Fig. 5.11a–d are taken with fixed powers and detunings, such that the additional cooling of one mode is accomplished just by varying the phases of the control tones. Since conventional laser cooling techniques (e.g., those using the single-tone dynamical backaction) are independent of these phases, the nonreciprocity demonstrated here can be used as a new resource for controlling the thermal fluctuations of a phononic oscillator.



high-order exceptional points

The concept of exceptional points is introduced in Ch. 1 as the degeneracy of an open (non-Hermitian) system. It is shown in Ch. 2 and Ch. 4 that encircling an EP in the parameter space can lead to nonreciprocal energy transfer within a two-level system. In the quasi adiabatic regime, such nonreciprocity depends on the topology of the encircling loop, as we have demonstrated with our optomechanical setup [141, 142]. For "trivial" adiabatic control loops that do not encircle an EP, there is no energy transfer so the system's evolution remains reciprocal.

It is natural then to think about EPs and the associated dynamics of an open system, consisting of more than two harmonic modes. From a mathematical point of view, EPs (and DPs) are singularities of a parameterized matrix. One may observe the emergence of such singularities by varying parameters that changes the elements of a matrix. The emergence corresponds to a sudden response of a system (e.g., its eigenvalues) to smooth changes of external conditions (e.g., the parameters), whose features are well studied in the catastrophe theory [152, 153].

Besides the dynamics of a system around the singularity, systems at the singularity itself are also of interest. For example, a matrix at DPs can be diagonalized because it can be regarded as a perturbation of some other hermitian matrix with distinct eigenvalues, and the corresponding eigenvectors remain linearly independent under any matrix perturbation¹. In contrast, a matrix at

^{1.} Note an Hermitian matrix with n distinct eigenvalues has n eigenvectors that are not only linearly independent but, more than that, orthogonal. By a continuity argument, one would see that the matrix perturbation may transform different eigenvalues to coincident ones, but it cannot make the orthogonal eigenvectors linearly dependent.

an EP is non-diagonalizable (defective), since the eigenvectors corresponding to the degenerate eigenvalues are parallel and therefore linearly dependent with each other. The effort to diagonalize a defective matrix will in general lead to a Jordan block (defined later), during which procedure a complete basis is formed by augmenting the eigenvectors with generalized eigenvectors.

This chapter covers basic theory of arbitrary-order singularities and our progress towards searching for the third order exceptional points (EP3). In Sec. 6.1, I consider general perturbations of a Jordan canonical form to answer the following question: how many parameters are necessary to guarantee the existence of a certain singularity? Equivalently: given a singularity, how many linearly independant ways are there to lift it? In Sec. 6.2, I study the space of second order EPs (EP2s) in the vicinity of an EP3. The effort to demonstrate the existence of EP3 on our optomechanical platform is discussed in Sec. 6.3. Then I conclude the chapter with future directions in Sec. 6.4.

6.1 Jordan canonical form and perturbations

It is well known in linear algebra that under a similarity transformation S, every $n \times n$ complex matrix M can be brought into Jordan canonical form:

$$M \to S M S^{-1} = \begin{pmatrix} J_1(\lambda_1) & 0 & 0 & 0 \\ 0 & J_2(\lambda_2) & 0 & 0 \\ & & & \\ 0 & 0 & 0 & J_k(\lambda_k) \end{pmatrix}$$
(6.1)

where $\lambda_1, \lambda_2,...\lambda_k$ are the eigenvalues of M with degeneracies $d_1, d_2,...d_k$, and the Jordan blocks J_i are $d_i \times d_i$ with λ_i on the major diagonal, 1 or 0 on the diagonal above the major diagonal, and 0 elsewhere. Consider general perturbations of a 2 × 2 Jordan block:

$$J_{2}' = J_{2} + \delta J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \delta J_{11} & \delta J_{12} \\ \delta J_{21} & \delta J_{22} \end{pmatrix}$$
(6.2)

where the eigenvalues of J_2 are set to be 0 without loss of generality². It turns out that a large collection of J' can be obtained from J_2 by performing a similarity transformation near the identity, i.e.,

$$J' = (I + \delta S)J_2(I + \delta S)^{-1} = J_2 + [\delta S, J_2] + O(\delta S^2)$$
(6.3)

We only keep the first order of δS on the right hand side since its elements are small. The perturbation generated by $\delta J = [\delta S, J_2]$ is trivial and can be neglected, as it leaves the eigenvalues unchanged and corresponds to no more than a rotation of axes. If we choose the small matrix δS to be:

$$\delta S = \begin{pmatrix} A & B \\ a & b \end{pmatrix} \text{ s.t. } [\delta S, J_2] = \begin{pmatrix} -a & A - b \\ 0 & a \end{pmatrix}$$
(6.4)

and compare with δJ , it is clear that the nontrivial perturbation must have the form:

$$\delta J = \begin{pmatrix} 0 & 0\\ \delta J_{21} & 0 \end{pmatrix} \tag{6.5}$$

This nontrivial perturbation of J_2 eliminates the degeneracy of eigenvalues, namely $\lambda(J_2) = 0$ and $\lambda(J') = \pm \sqrt{\delta J_{21}}$. Conversely, for a general 2×2 complex matrix M with distinct eigenvalues, we may vary one off-diagonal element M_{21} to make the eigenvalues coincide. The fact that M_{21} being a complex number indicates that we need two independently tunable real parameters to guarantee the existence of an EP.

We have demonstrated the existence of an EP in our optomechanical setup, with the two parameters chosen to be laser power and detuning. Combine with the discussion in previous paragraph, having two real parameters are necessary and sufficient to realize EP2 in a coupled two-mode system.

Similar treatment can be applied to 3×3 matrices to gain some insight onto EP3. The perturbed matrix is:

$$M' = M + \delta M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \delta M_{11} & \delta M_{12} & \delta M_{13} \\ \delta M_{21} & \delta M_{22} & \delta M_{23} \\ \delta M_{31} & \delta M_{32} & \delta M_{33} \end{pmatrix}$$
(6.6)

The trivial perturbation, which can be written as the commutator of some small matrix S with M,

^{2.} This is because the spectrum of a matrix can be shifted arbitrarily, which corresponds to adding $\lambda I_{n \times n}$ to the matrix.

has the form:

$$\delta M = \begin{pmatrix} -a & A-b & B-c \\ -\alpha & a-\beta & b-\gamma \\ 0 & \alpha & \beta \end{pmatrix} \text{ where } S = \begin{pmatrix} A & B & C \\ a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}$$
(6.7)

The general nontrivial perturbation δM should obey $(\delta M)_{31} \neq 0$ and $(\delta M)_{21} \neq -(\delta M)_{32}$, thus can be written as:

$$\delta M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix}$$
(6.8)

where *x*, *y* are independent complex numbers. Therefore, a general three-mode open system with four linearly independent real parameters can be tuned to an EP3. Furthermore, one can show that 2(N - 1) real parameters are necessary to realize an Nth order EP for a general *N*-mode open system.

It is worthwhile to draw a comparison with DPs — the counterpart of EPs in closed (Hermitian) systems. The perturbations are added on a trivial matrix (i.e., a matrix with all elements zero), instead of on a Jordan block in the non-Hermitian case. Therefore to get an n^{th} order DP, we need to control n(n-1)/2 off-diagonal complex numbers as well as (n-1) diagonal real numbers (not *n* as the matrix is assumed to be traceless), that is $(N^2 - 1)$ real parameters in total. Roughly speaking, given a large number *n* (the order of degeneracy), it is easier to access EPs than DPs, since less parameters are required to be changed. In this sense, EPs are more general and stable.

6.2 EP2 space near EP3

In this section we consider another question: if a system originally sitting at an EP3 is slightly perturbed, what kind of perturbation would leave the system at some EP2?

According to the previous section, the perturbed Hamiltonian of the system can be written (directly or after a similarity transformation) as:

$$H(x,y) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & y & 0 \end{pmatrix}$$
(6.9)



Figure 6.1: Knot structure of EP2 subspace. For illustration purpose, the parameters are set numerically and have arbitrary units.

The eigenvalues of H(x, y) are denoted as λ_i (i = 1, 2, 3), which are roots of the characteristic polynomial $f(\lambda) \equiv -\lambda^3 + y\lambda + x$.

To characterize the degeneracy of eigenvalues, it is convenient to introduce the discriminant of the characteristic polynomial. In general, for a polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$ with roots λ_i , one can define $\text{Disc}(P) = (-1)^{n(n-1)/2} a_n^{2n-2} \prod_{i < j} (\lambda_i - \lambda_j)$, which is zero if and only if *P* has at least one multiple root. Specifically, $\text{Disc}(f) = 4y^3 - 27x^2 = 0$ for $|x|^2 + |y|^2 > 0$ if and only if the system is at EP2³.

The parameter space of the system is four dimensional since $x, y \in \mathbb{C}$, and the EP2 subspace characterized by Disc(f) = 0 is a two dimensional structure. Since it is unintuitive to imagine a coordinate system with more than three axes, we may cut a "slice" from the four dimensional parameter space and study the EP2 structure enclosed in this slice. A natural slice would be a 3-sphere, described by

$$|x|^{2} + |y|^{2} = r_{x}^{2} + r_{y}^{2} = r_{0}^{2}$$
(6.10)

where $x = r_x e^{i\theta_x}$, $y = r_y e^{i\theta_y}$ $(r_x, \theta_x, r_y, \theta_y \in \mathbb{R}^+)$ and r_0 is some fixed small real number. The parameters of an EP2 on this 3-sphere also satisfies:

$$\operatorname{Disc}(f) = 0 \iff 27r_x^2 = 4r_y^3, \theta_x = \frac{3}{2}\theta_y \tag{6.11}$$

^{3.} The system is at EP3 when x = y = 0.

Combining Eq. 6.10 and Eq. 6.11, we can solve $r_{x,y}$ as functions of r_0 , and the constraint on $\theta_{x,y}$ specifies a trefoil knot, which lies on the surface of some torus. The EP2 subspace on the 3-sphere is visualized in Fig. 6.2.

Recall that the EP subspace of a two-mode systems is a single point in 2D parameter space. The associated topological dynamics for a parameter loop is simple: the loop may either encircle the EP2 or not. In three-mode systems, the topological dynamics can be much more complicated. A parameter loop can be chosen to "encircle" the EP2 space (the knot in Fig. 6.2 in four topologically distinct ways. (A loop cannot "encircle" EP3 in this space.)

6.3 Optomechanical EP3

We now examine how an EP3 can be demonstrated with our optomechanical setup. Three mechanical modes can be selected and coupled with each other via cavity mediated optomechanical interaction. A naive approach would be to use a straightforward generalization of Ref. [141], where three modes of a nearly degenerate triplet (e.g., $\{1, 7\}$, $\{7, 1\}$, $\{5, 5\}$) are coupled. These mechanical modes are indeed coupled together as long as a laser is sent into the cavity. However, one can show that the couplings between the three modes are linearly dependent on each other⁴, so an EP3 can not be reached in this way.

Alternatively, we may consider to generalize the concept of the VEP [142], and to demonstrate EP3 of the dynamical matrix in some rotating frame (while the modes are non-degenerate in the lab frame). The idea is to select three non-degenerate mechanical modes $\omega_{m,i}$ (i = 1, 2, 3), and apply three control laser tones with detunings $\Delta_i \approx -\omega_{m,i}$. The mechanical sidebands added on lasers will be close to the cavity resonance, and the beat notes between these sidebands and the laser tones may drive the mechanical modes, thereby coupling the modes effectively.

^{4.} In fact, the inter-mode couplins have the same form as the off-diagonal terms of H in Ref. [141]. They will be proportional to the laser power, and is not sensitive to changes in the detuning.



Figure 6.2: Possible setups to search for EP3s. Left: a single control laser tone to couple a near-degenerate triplet. Right: three control laser tones to couple three non-degenerate modes.

The dynamical matrix of such a system can be derived following the same procedure as in Ch. 4,

$$D(t) = \begin{pmatrix} \omega_1 - i\frac{\gamma_1}{2} + \sigma_{11} & \sigma_{12}e^{i\Delta_{12}t} & \sigma_{13}e^{i\Delta_{13}t} \\ \sigma_{21}e^{i\Delta_{21}t} & \omega_2 - i\frac{\gamma_2}{2} + \sigma_{22} & \sigma_{23}e^{i\Delta_{23}t} \\ \sigma_{31}e^{i\Delta_{31}t} & \sigma_{32}e^{i\Delta_{32}t} & \omega_3 - i\frac{\gamma_3}{2} + \sigma_{33} \end{pmatrix}$$
(6.12)

where $\sigma_{mn} = ig_m g_n \alpha_m^* \alpha_n (\chi (\omega_m - \Delta_n) - \chi (\omega_m + \Delta_m))$ and $\Delta_{ij} = \Delta_i - \Delta_j$. Define the rotating matrix

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\Delta_{12}t} & 0 \\ 0 & 0 & e^{-i\Delta_{13}t} \end{pmatrix}$$
(6.13)

which transforms the mode amplitude vector $c(t) = (c_1(t), c_2(t), c_3(t))^T$ into c'(t) = Uc(t), we end up with a time-independent dynamical matrix written as:

$$D^{r} = \begin{pmatrix} \omega_{1} - i\frac{\gamma_{1}}{2} + \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \omega_{2} - \Delta_{12} - i\frac{\gamma_{2}}{2} + \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \omega_{3} - \Delta_{13} - i\frac{\gamma_{3}}{2} + \sigma_{33} \end{pmatrix}$$
(6.14)

such that the equation of motion is $i\dot{c}'(t) = D^{r}c'(t)$. We are able to access an EP3 of D^{r} as long as σ_{ij} can be tuned linearly independently. For the setup with three control laser beams, there are six parameters in total (the power and detuning of each tone), which is more than the minimal theoretical requirement. Experimentally, one can set control laser powers as three free parameters and choose some common detuning (relative to the bare mechanical mode) as the fourth parameter. For $\{1, 2\}\{2, 1\}$ mode pair and $\{2, 2\}$ singlet mode, numerical simulation shows that a triplet degeneracy emerges when lasers are ~10 μ W and the common detuning tens of kHz⁵.

Yet another scheme to explore EP3 is to use one laser tone to create an EP for a pair of nearlydegenerate mechanical modes, and then add a second laser tone to bring a third mechanical mode into effective degeneracy with the other two. This scheme may be treated as a special case of the above proposal. Assume $\omega_1 \approx \omega_2$, the two tones detuned at Δ_1, Δ_2 in above proposal are equivalent to the first tone in current scheme. Specifically, we have $\Delta_{12} = 0$, $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{23}$, $\sigma_{31} = \sigma_{32}$. Note the number of parameters is reduced to four in this scheme, which is just enough to access an EP3. Numerical simulation shows that (again for $\{1, 2\}\{2, 1\}$ mode pair and $\{2, 2\}$ singlet mode) an EP3 can be accessed with $P_1 \approx 150 \ \mu$ W, $\Delta_1 \approx -2\pi \times 500 \ k$ Hz, $P_2 = 50 \ \mu$ W, $\Delta_2 \approx -2\pi \times 600 \ k$ Hz. In practice, the optical damping effect makes the measurement of the modes' oscillations quite challenging.

Experimentally, the eigenvalues of D^r are extracted from the mechanical driven measurements described in Ch. 4, with fitting functions derived in App. A. Note that the spectrum near each center frequency has three peaks, instead of two in the previous two-mode case. These measurements are were begun during this thesis writing, and we expect to have some result in the near future.

6.4 Future directions

A lot of static and dynamic behaviors can be studied once we demonstrate an EP3 of our system. It will be interesting to show the knot structure of EP2s nearby the EP3. We may also observe the evolution of the system along various parameter loops to study the (non)reciprocal as well as topological features of such energy transfer. As pointed out in a recent theoretical study, that the geometric phase for adiabatic closed-loop operation within the degenerate EP space is topological and it is determined by the homotopy class of the control loop [154].

Meanwhile, a potential application of EPs in sensing technique has been proposed recently [155]. The idea is based on the high sensitivity of the eigenvalues to perturbations in coupling parameters

^{5.} For numerical simulation, we consider the Jordan normal form for D^r and numerically find the solution of $x = \text{Det}(D^r) = 0$ and $y = \text{Tr}(D^r \cdot D^r)/2 = 0$. We do not present the exact numerical values here, as they are very sensitive to the optomechanical coupling coefficients, which are subject to change over months in experiments.

(e.g., near an EP*n*, a parameter perturbation ϵ leads to $\epsilon^{1/n-1}$ change of the eigenvalues, which is considerably largger than the linear term ϵ for perturbation at regular points). Following this path, enhanced mode splitting via utilizing exceptional points has been experimentally demonstrated in microtoroid cavities [110,111]. We can explore this sensitivity in our setup, perhaps as a transducer of the quantum fluctuations of the cavity photons. Nonreciprocity is shown to be useful in sensing, as it allows one to exceed the fundamental bounds constraining any conventional, reciprocal sensor [156]. It is worth mentioning that noises should be treated properly when characterizing the EP-based sensors [157, 158].

There has been a growing interest in the study of high-order EPs in other systems as well. Existence of high-order EPs has been demonstrated in various systems such as ultracold Bose gases [159], coupled photonic resonators [160] and cavity magnons [161].



Concluding remarks

This dissertation reviews the work in HarrisLab that demonstrates robust nonreciprocal interaction between two phononic resonators. The nonreciprocity, either transient or static, is realized via the cavity mediated optomechanical interaction, and can be noticed by the asymmetry between off-diagonal elements of the modes' dynamical matrix. If the mechanical modes are coupled via two control laser tones (Ch. 4), only the phases of these elements are different, and the nonreciprocal energy transfer is investigated in the presence of a VEP. With two additional tones (Ch. 5), both amplitudes and phases of these elements differ from each other, and we have demonstrated the isolation in phonon transmission and also the ability to control the thermal fluctuations.

The study of nonreciprocity is an active field. Optomechanical systems are shown to be a promising candidate for realizing on-chip nonreciprocal devices, yet it is hard to draw a definite comparison with the current commercial devices, as the desired metrics of performance may vary with the application, and optomechanical approaches are still in their infancy. Several crucial parameters such as bandwidth, linearity, power consumption and noise should be characterized as these devices continue to be improved.

Other intriguing possibilities in optomechanical nonreciprocity include realizing topological insulators for photons and phonons [162]. Coupling together large networks of optomechanical resonators may provide a natural integrated, compact platform to realize largely reconfigurable unidirectional transport for sound and light.

I hope this dissertation has provided a clear introduction to the field of optomechanical nonreciprocity, as well as a motivation to further exploration on high-order EPs in optomechanical systems.



Additional theoretical derivations

A.1 A note on nonreciprocity of EP encircling

Dynamically encircling an EP can be mapped to a two-mode scattering problem in a microwave waveguide. The full scattering matrix is symmetric, meaning it is reciprocal. In our system (coupled nearly-degenerate mechanical modes), it is guaranteed by the dynamical matrix being symmetric.

$$U_{\cup}(\tau, 0) = e^{-i\int_{0}^{\tau} D(r_{\cup}(t))dt} = e^{-i\int_{0}^{\tau} D(r_{\cup}(\tau-t))dt}$$

$$= [e^{-i\int_{0}^{\tau} D^{\mathrm{T}}(r_{\cup}(t))dt}]^{\mathrm{T}} = [e^{-i\int_{0}^{\tau} D(r_{\cup}(t))dt}]^{\mathrm{T}} = U_{\cup}^{\mathrm{T}}(\tau, 0)$$
(A.1)

where we have already used $D^{T} = D$. Note that according to matrix multiplication rule $(AB) = (B^{T}A^{T})^{T}$, the change of order in the integral is associated with the transposing the matrices.

A.2 Heterodyne measurement signal

In general, an optical field $\hat{a}_{out}(t)$ landing on a photodetector creates a photocurrent $I(t) = \sigma G \langle \hat{N}(t) \rangle$, where $\hat{N}(t) = \hat{a}_{out}^{\dagger}(t)\hat{a}_{out}(t)$ is the photon number operator, σ is the quantum efficiency of the detector and *G* is the gain. Introduce the current autocorrelation function:

$$\overline{i(t)i(t+\tau)} = G^2 \sigma^2 \langle : \hat{N}(t)\hat{N}(t+\tau) : \rangle + \sigma \langle \hat{N}(t) \rangle \delta(t)$$
(A.2)

where :: indicates normal and time ordering. The first term is the time-ordered correlation of doublephoton counting, and the second term is the autocorrelation of single-photon counting. Note that we have assumed the detector to have an infinite bandwidth in Eq. (A.2). For a finite-bandwidth photodetector, we visualize the photon counting events as generating photoelectric pulses within a nonzero duration τ_d . A single photon arrived in $[t, t + \tau]$ will contribute to both i(t) and $i(t + \tau)$, thereby adding a nonzero contribution to $S[\omega]$.

The power spectrum of the photocurrent can be written as:

$$S[\omega] = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_{-\infty}^{\infty} d\tau \overline{i(t)i(t+\tau)}$$
(A.3)

To calculate the heterodyne signal in our measurement, consider the input mode

$$\hat{a}_{\rm in}(t) = e^{-i\Omega t} \left[K(1 + \sqrt{r}e^{i(\omega_{if}t+\theta)}) + \hat{\xi}(t) \right] \tag{A.4}$$

where we have included the local oscillator beam at $\Omega - \omega_{if}$, and $r = P_{lo}/P_{meas} \gg 1$ is the ratio between the LO and the measurement beam. $K = \sqrt{P/\hbar\Omega}$, and θ is the phase of the spectrum. Note we have neglected the classical noise of lasers. According to input-output theory, the output mode is

$$\hat{a}_{\text{out}}(t) = \hat{a}_{\text{in}}(t) - \sqrt{\kappa_{\text{in}}}\hat{a}(t) \tag{A.5}$$

where $\hat{a}(t)$ is the intra-cavity field that can be linearized into an average amplitude and a small fluctuating term. Namely,

$$\hat{a}(t) = e^{-i\Omega t} (\alpha + \hat{d}(t))$$
 where $\alpha = \frac{\sqrt{\kappa_{in}}K}{\kappa/2 - i\Delta}$ (A.6)

By solving \hat{d} in the equations of motion for the optomechanical system, we can use the above equations to derive $S[\omega]$, which includes both red and blue sidebands.

A.3 PDH error signal

Pound-Drever-Hall (PDH) technique is generally used to keep a laser and a Fabry-Perot cavity mode on resonance. Here we describe how the error signal is generated.

In the PDH setup, the laser beam is phase modulated before it enters the cavity, which is usually done by an electro-optic modulator (EOM). Suppose the laser generates a coherent electromagnetic field with amplitude E_0 at $\omega_L/2\pi$, and the EOM provides phase modulation at frequency $\Omega/2\pi$ with a small modulation depth β , the electric field incident on the cavity input mirror is:

$$E_{\rm inc} = E_0 e^{-i(\omega_L t + \beta \sin \Omega t)}$$

$$\simeq E_0 [J_0(\beta) - 2iJ_1(\beta) \sin \Omega t] e^{-i\omega_L t}$$
(A.7)

$$= E_0 [J_0(\beta) e^{-i\omega_L t} + J_1(\beta) e^{-i(\omega_L + \Omega)t} - J_1(\beta) e^{-i(\omega_L - \Omega)t}]$$

where Js are Bessel functions. As we can see, there are three different beams: a carrier with frequency $\omega_L/2\pi$ and two sidebands with frequencies $(\omega_L \pm \Omega)/2\pi$. If P_0 is the total power in the beam, then the power in the carrier and sidebands are $P_c = J_0^2(\beta)P_0$ and $P_s = J_1^2(\beta)P_0$, separately.

For a single incident field $E_{inc} = E_0 e^{-i\omega t}$, the reflected beam is the coherent sum of two different beams: the promptly reflected beam off the front mirror, and the leakage beam from the cavity (which contains multiple components that bounces between two mirrors and transmitted through the front mirror). Let r_1 (r_2) denote the amplitude reflection coefficient for the front (back) mirror, and $t_1 = \sqrt{1 - r_1^2}$ is the transmission coefficient for the front mirror. The total reflected beam is:

$$E_{\text{ref}} = E_0[-r_1 e^{-i\omega t} + \sum_{k=1}^{\infty} t_1 r_2^k r_1^{k-1} t_1 e^{-i\omega(t+2kL/c)}]$$
(A.8)

so the total reflection coefficient is:

$$F(\omega) = \frac{E_{\text{ref}}}{E_{\text{inc}}} = \frac{r_2 e^{-i\omega/\Delta v_{\text{FSR}}} - r_1}{1 - r_1 r_2 e^{-i\omega/\Delta v_{\text{FSR}}}}$$
(A.9)

where $\Delta v_{\text{FSR}} = c/2L$ is the free spectral range of the cavity.



Figure A.1: PDH error signal (a.u.) as a function of detuning. The phase modulation frequency $\Omega \simeq 0.004\omega_c$, the mirror reflection coefficients are $r_1 = 0.9998$ and $r_2 = 0.99997$, respectively.

To calculate the reflection when there are multiple incident beams, we multiply each incident beam with the reflection coefficient at beam frequency. The reflected beam in PDH setup is therefore:

$$E_{\text{ref}} = E_0 e^{-i\omega_L t} [F(\omega_L) J_0(\beta) + F(\omega_L + \Omega) J_1(\beta) e^{-i\Omega t} - F(\omega_L - \Omega) J_1(\beta) e^{i\Omega t}]$$
(A.10)

The beam is measured with a photodetector, which yields:

$$P_{\text{ref}} = |E_{\text{ref}}|^2 = P_c |F(\omega_L)|^2 + P_s [|F(\omega_L + \Omega)|^2 + |F(\omega_L - \Omega)|^2] + 2\sqrt{P_c P_s} \{\Re[F(\omega_L)F^*(\omega_L + \Omega) - F^*(\omega_L)F(\omega_L - \Omega)]\cos\Omega t$$
(A.11)
$$-\Im[F(\omega_L)F^*(\omega_L + \Omega) - F^*(\omega_L)F(\omega_L - \Omega)]\sin\Omega t\}$$

where we have neglected 2Ω and higher order terms. The detected signal is mixed with the EOM driving signal, which is proportional to $\sin \Omega t$, and the resulting error signal is:

$$\varepsilon = 2\sqrt{P_c P_s} \Im[F^*(\omega_L)F(\omega_L - \Omega) - F(\omega_L)F^*(\omega_L + \Omega)]$$
(A.12)

Fig. A.1 shows our measured error signal as a function of laser detuning $\Delta = \omega_L - \omega_c$, where $\omega_c = n\Delta v_{FSR}$ for some positive integer mode number *n* is a cavity resonance frequency. We can see the error signal is linear within a small range near ω_c , and can be used in the feedback loop to

stabilized Δ around 0.

A.4 Driven response measurement of three modes

Write down the equations of motion in the Fourier domain:

$$\boldsymbol{c}'\left[\omega\right] = \left(\omega I_{3\times 3} - \boldsymbol{D}^{\mathrm{r}}\right)^{-1} \boldsymbol{F}\left[\omega\right] \tag{A.13}$$

which indicates the spectrum near each center frequency has three peaks. To see this explicitly, we consider $\omega \approx \omega_{m,1}$. The main contribution to the photocurrent comes from c_1 that yield the form:

$$c_1[\omega] = \sum_{i=1}^3 \frac{h_{1i}}{\omega - \tilde{\omega}_i} F_1[\omega]$$
(A.14)

where $\tilde{\omega}_i$ (*i* = 1, 2, 3) is the eigenvalue of D^r and h_{1i} is the corresponding complex height. Suppose the diagonal elements of D^r are denoted as D_1 , D_2 and D_3 and let

$$A = -(D_1 + D_2 + D_3)$$

$$B = D_1 D_2 + D_2 D_3 + D_3 D_1$$
(A.15)

Compare two expressions Eq.A.13 and Eq.A.14 of c_1 , we have

$$(\tilde{\omega}_{2} + \tilde{\omega}_{3}) h_{11} + (\tilde{\omega}_{1} + \tilde{\omega}_{3}) h_{12} + (\tilde{\omega}_{1} + \tilde{\omega}_{2}) h_{13} = D_{2} + D_{3}$$
$$\tilde{\omega}_{2} \tilde{\omega}_{3} h_{11} + \tilde{\omega}_{3} \tilde{\omega}_{1} h_{12} + \tilde{\omega}_{1} \tilde{\omega}_{2} h_{13} = D_{2} D_{3} - \sigma_{23} \sigma_{32}$$
$$(A.16)$$
$$h_{11} + h_{12} + h_{13} = 1$$

Now set $h_{1i} = \frac{1}{3} + x_i$, we derive:

$$\sum_{i=1}^{3} \left(\tilde{\omega}_{(i+1)\%3} + \tilde{\omega}_{(i+2)\%3} \right) x_i = D_2 + D_3 + \frac{2}{3}A$$

$$\sum_{i=1}^{3} \tilde{\omega}_{(i+1)\%3} \tilde{\omega}_{(i+2)\%3} x_i = D_2 D_3 - \sigma_{23} \sigma_{32} - \frac{1}{3}B$$

$$\sum_{i=1}^{3} x_i = 0$$
(A.17)

from where we can solve

$$x_{1} = \frac{\tilde{\omega}_{1} \left(D_{2} + D_{3} + \frac{2}{3}A \right) - D_{2}D_{3} + \sigma_{23}\sigma_{32} + \frac{1}{3}B}{\left(\tilde{\omega}_{1} - \tilde{\omega}_{2} \right)\left(\tilde{\omega}_{3} - \tilde{\omega}_{1} \right)}$$
(A.18)

$$x_{2} = \frac{\tilde{\omega}_{2} \left(D_{2} + D_{3} + \frac{2}{3} A \right) - D_{2} D_{3} + \sigma_{23} \sigma_{32} + \frac{1}{3} B}{(\tilde{\omega}_{1} - \tilde{\omega}_{2}) (\tilde{\omega}_{2} - \tilde{\omega}_{3})}$$
(A.19)

$$x_{3} = \frac{\tilde{\omega}_{3} \left(D_{2} + D_{3} + \frac{2}{3} A \right) - D_{2} D_{3} + \sigma_{23} \sigma_{32} + \frac{1}{3} B}{(\tilde{\omega}_{2} - \tilde{\omega}_{3}) (\tilde{\omega}_{3} - \tilde{\omega}_{1})}$$
(A.20)

Note that when $\sigma_{ij} = 0$ $(j \neq i)$ we have $h_{11} = 1$ and $h_{12} = h_{13} = 0$, which means the first peak is the real (or physical) peak and the other two are "ghost" peaks.

Similarly for the other two modes, we write:

$$c_{2}[\omega] = \sum_{i=1}^{3} \frac{h_{2i}}{\omega - \Delta_{12} - \tilde{\omega}_{i}} F_{2}[\omega]$$
(A.21)

$$c_{3}[\omega] = \sum_{i=1}^{3} \frac{h_{3i}}{\omega - \Delta_{13} - \tilde{\omega}_{i}} F_{3}[\omega]$$
(A.22)

Set $h_{2i} = \frac{1}{3} + y_i$ and $h_{3i} = \frac{1}{3} + z_i$ such that:

$$y_{i} = \frac{\tilde{\omega}_{i} \left(D_{1} + D_{3} + \frac{2}{3}A \right) - D_{1}D_{3} + \sigma_{13}\sigma_{31} + \frac{1}{3}B}{\left(\tilde{\omega}_{i} - \tilde{\omega}_{(i+1)\%3} \right) \left(\tilde{\omega}_{(i+2)\%3} - \tilde{\omega}_{i} \right)}$$
(A.23)

$$z_{i} = \frac{\tilde{\omega}_{i} \left(D_{1} + D_{2} + \frac{2}{3}A \right) - D_{1}D_{2} + \sigma_{12}\sigma_{21} + \frac{1}{3}B}{\left(\tilde{\omega}_{i} - \tilde{\omega}_{(i+1)\%3} \right) \left(\tilde{\omega}_{(i+2)\%3} - \tilde{\omega}_{i} \right)}$$
(A.24)

To summarize, mechanical driven response is:

$$c_{j}[\omega] = \sum_{i=1}^{3} \frac{h_{ji}}{\omega - \Delta_{1j} - \tilde{\omega}_{i}} F_{j}[\omega]$$
(A.25)

where

$$h_{ji} = \frac{\tilde{\omega}_i \left(D_{j+1} + D_{j+2} + \frac{2}{3}A \right) - D_{j+1}D_{j+2} + \sigma_{(j+1)(j+2)}\sigma_{(j+2)(j+1)} + \frac{1}{3}B}{(\tilde{\omega}_i - \tilde{\omega}_{i+1})(\tilde{\omega}_{i+2} - \tilde{\omega}_i)}$$
(A.26)

where all subscriptions are in mod3 sense such that 4=1 and 5=2 (3=3, not 0).



Additional experimental details

B.1 Characterization of the system

B.1.1 Linewidth of the mechanical modes

The bare linewidth of a mechanical mode is characterized by the membrane ring-down measurement. Specifically, the mode is first driven to some large amplitude via a sinusoidal wave that is sent to the membrane piezo, and then oscillates freely without the external drive.



Figure B.1: Examples of mechanical ringdowns for the {1, 1}, {2, 2}, and {3, 3} modes at the corresponding frequencies, plotted in red, blue, and green, respectively. The black lines are fits of the decaying amplitude to $\sqrt{a^2 e^{-\gamma t} + b^2}$, where *a* is the initial amplitude of the motion and *b* is a background.

We use a 1310 nm laser (manufactured by ThorLabs) instead of the 1064 nm measurement laser to measure the motion in the mode. The reasons are(i) the optical damping effect caused by a near-resonance 1064 nm laser perturbs the mechanical linewidth significantly even when the beam is weak, and (ii) it is difficult to lock the 1064 nm beam to the cavity when the membrane oscillates with a large amplitude. The 1310 nm beam can enter the cavity without any need for locking since the finesse of the cavity at 1310 nm is close to unity. Moreover, as the finesse is so low, the optical damping is negligible.

Time traces of such measurement on three different mechanical modes are shown in Fig. B.1. The bare linewidth of each mode can be extracted from the fitting result.

B.1.2 Linewidth of the optical cavity

To characterize κ and κ_{in} of the optical cavity, we sweep the frequency of the measurement laser over the cavity resonance and record the DC reflection spectrum and the PDH error signal on an oscilloscope (shown in Fig. B.2). The error signal is used to calibrate the time axis of the sweep into unit of Hz, and the reflection voltage signal near cavity resonance is fit with:

$$V(\Delta) = G|1 - \frac{\kappa_{\rm in}}{\kappa/2 + i(\Delta - \Delta_0)}|^2 + b \tag{B.1}$$

1.0 0.8 0.6 0.4 0.2 -2000 -1800 -1600 -1400 -1200 -1000 -800 Frequency (kHz)

where G is the overall gain, b is the background, Δ_0 is the detuning offset.

Figure B.2: Cavity linewidth measurement via reflection.

B.1.3 Optomechanical coupling rates

The optomechanical coupling rates g_a for mode *a* can be extracted via fitting to the optical spring and damping measurement result. As illustrated in Fig. B.3.



Figure B.3: Examples of dynamical backaction fitting for the $\{2, 1\}$ and $\{2, 2\}$ modes. The control beam is 14 μ W with detuning varied over a large range. The optical spring and damping data is from the fitting to the brownian motion sidebands.

B.2 Initialization of the experiment

The optical alignment and the cool down of the cryogenic platform was already completed when I joined Harris lab in March 2015. The initialization I discuss here refers to locking the lasers in order, and relocating the membrane position every time after the helium transfer¹.

The preparation for a measurement is as follows. We first bypass the filter cavities and lock the probe beam to cavity (with PI controller No. 1&2). Then we lock the control beam to the probe (with PI controller No. 3). At this point we can locate the membrane to its optimal position or "sweet spot" (discussed below). We then lock the filter cavities (with PI controller No. 4&5). The steps to do that are (i) break all prior lockings, (ii) lock measurement laser filter cavity, (iii) relock the probe and control lasers, and (iv) lock the control laser filter cavity.

The sweet spot refers to the membrane position where the frequency separation between two cavity modes (addressed by control and measurement laser, respectively) is maximized. To measure

^{1.} The cryostat is de-floated before transferring helium, so the position of membrane in the cavity may shift after re-floating by the end of the transfer.

this frequency separation, we sweep the frequency of the Rohde&Schwarz signal generator (that produces the ≈ 8 GHz reference signal) with a deviation of several MHz at a frequency of ≈ 2 Hz. The control laser lock tracks the frequency sweep of reference signal, so the frequency of the control laser experience the same sweep. As control laser frequency passes over the cavity mode, a dip in reflected light is observed. We then adjust the center frequency of the sweep until the dip is in the middle of the whole sweeping range. (A coarse way to verify that is to observe equal-distant reflection dips on an oscilloscope.) Now the center frequency is a good approximation to the mode separation, which we maximize by adjusting the membrane position.²

^{2.} Note that this method relies on the fact that the probe beam is exactly on resonance with the cavity mode it addressed, which may not be the case. We usually keep the probe beam slightly red-detuned to self-oscillations and unlocking of our system.

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